General Modelling and Scaling Laws

- Dimensionless numbers
- Similarity requirements
- Derivation of dimensionless numbers used in model testing
- Froude scaling
- Hydroelasticity
- Cavitation number
Dimensionless numbers

• "Without dimensionless numbers, experimental progress in fluid mechanics would have been almost nil; It would have been swamped by masses of accumulated data” (R. Olson)

• **Example:**
Due to the beauty of dimensionless numbers, $C_f$ of a flat, smooth plate is a function of $Re$ only (not function of temperature, pressure or type of fluid\(^1\))

\(^1\)As long as the fluid is Newtonian, which means that it has a linear stress/strain rate, with zero stress for zero strain
Fig. 2  Skin friction lines, turbulent and laminar flow

PRANDTL-VON KARMAN  $C_F = 0.072 \left( \frac{V}{L} \right)^{-\frac{1}{3}}$

BLASIUS  $C_F = 1.327 \left( \frac{V}{L} \right)^{-\frac{1}{2}}$
Types of similarity

- Geometrical similarity
- Kinematic similarity
- Dynamic similarity

What are the similarity requirements for a model test?
Geometrical Similarity

- The model and full scale structures must have the same shape
  \[ \Rightarrow \text{All linear dimensions must have the same scale ratio:} \quad \lambda = \frac{L_F}{L_M} \]

- This applies also to:
  - The environment surrounding the model and ship
  - Elastic deformations of the model and ship
Kinematic Similarity

• Similarity of velocities:

⇒ The flow and model(s) will have geometrically similar motions in model and full scale

Examples:
- Velocities in x and y direction must have the same ratio, so that a circular motion in full scale must be a circular motion also in model scale

- The ratio between propeller tip speed and advance speed must be the same in model and full scale:

\[
\frac{V_F}{n_F (2\pi R_F)} = \frac{V_M}{n_M (2\pi R_M)} \quad \text{or} \quad \frac{V_F}{n_F D_F} = \frac{V_M}{n_M D_M} \Rightarrow J_F = J_M
\]
Dynamic Similarity

• Geometric similarity and
• Similarity of forces
  ⇒ *Ratios between different forces in full scale must be the same in model scale*
  ⇒ *If you have geometric and dynamic similarity, you’ll also have kinematic similarity*
• The following force contributions are of importance:
  – Inertia Forces, $F_i$
  – Viscous forces, $F_v$
  – Gravitational forces, $F_g$
  – Pressure forces, $F_p$
  – Elastic forces in the fluid (compressibility), $F_e$.
  – Surface forces, $F_s$. 
Inertia Forces (mass forces)

\[ F_i \propto \rho \frac{dU}{dt} L^3 = \rho \frac{dU}{dx} \frac{dx}{dt} L^3 \propto \rho U^2 L^2 \]

- \( \rho \) is fluid density
- \( U \) is a characteristic velocity
- \( t \) is time
- \( L \) is a characteristic length (linear dimension)
Gravitational Forces

\[ F_g \propto \rho g L^3 \]

⇒ Just mass times acceleration
• \( g \) is acceleration of gravity
Viscous Forces

\[ F_v \propto \mu \frac{dU}{dx} L^2 \propto \mu UL \]

- \( \mu \) is dynamic viscosity [kg/m\( \cdot \)s]
  - a function of temperature and type of fluid
Pressure Forces

\[ F_p \propto pL^2 \]

⇒ Force equals pressure times area

• \( p \) is pressure
Elastic Fluid Forces

\[ F_e \propto \varepsilon_v E_v L^2 \]

- \( \varepsilon_v \) is compression ratio
- \( E_v \) is the volume elasticity (or compressibility)
- \( \varepsilon_v \cdot E_v \) = elasticity modulus \( K \) [kg/m \cdot s^2]
Surface Forces

\[ F_s \propto \sigma L \]

- \( \sigma \) is the surface tension [kg/s\(^2\)]
Froude number $Fn$

- The ratio between inertia and gravity:

$$\frac{\text{Inertia force}}{\text{Gravity force}} = \frac{F_i}{F_g} \propto \frac{\rho U^2 L^2}{\rho g L^3} = \frac{U^2}{gL}$$

- Dynamic similarity requirement between model and full scale:

$$\frac{U_M^2}{gL_M} = \frac{U_F^2}{gL_F}$$

$$\frac{U_M}{\sqrt{gL_M}} = \frac{U_F}{\sqrt{gL_F}} = Fn$$

- Equality in $Fn$ in model and full scale will ensure that gravity forces are correctly scaled

- Surface waves are gravity-driven $\Rightarrow$ equality in $Fn$ will ensure that wave resistance and other wave forces are correctly scaled
Reynolds number $Re$

- Equal ratio between inertia and viscous forces:

$$\frac{\text{Inertia forces}}{\text{Viscous forces}} = \frac{F_i}{F_v} \propto \frac{\rho U^2 L^2}{\mu UL} = \frac{\rho UL}{\mu} = \frac{UL}{\nu} = Re$$

- $\nu$ is the kinematic viscosity, $\nu = \frac{\mu}{\rho}$ [m$^2$/s]

- Equality in $Re$ will ensure that viscous forces are correctly scaled
To obtain equality of both $F_n$ and $R_n$ for a ship model in scale 1:10: $\nu_m = 3.5 \times 10^{-8}$
Mach number $M_n$

- Equal ratio between inertia and elastic fluid forces:
  \[
  \frac{F_i}{F_e} \propto \frac{\rho U^2 L^2}{\varepsilon_v E_v L^2}
  \]

- By requiring $\varepsilon_v$ to be equal in model and full scale:
  \[
  \left( \frac{\rho U^2 L^2}{\varepsilon_v E_v L^2} \right)_M = \left( \frac{\rho U^2 L^2}{\varepsilon_v E_v L^2} \right)_F
  \]
  \[
  \frac{U_M}{\sqrt{E_{v,M}/\rho}} = \frac{U_F}{\sqrt{E_{v,F}/\rho}} = M_n
  \]

- $\sqrt{E_v/\rho}$ is the speed of sound

- Fluid elasticity is very small in water, so usually Mach number similarity is not required
  - It is only when Mach numbers get close to 1 that it is important to consider compressibility effects. When Mach<0.7, incompressible flow is assumed
Weber number $W_n$

- The ratio between inertia and surface tension forces:

\[
\frac{\text{Inertia forces}}{\text{Surface tension forces}} = \frac{F_i}{F_s} \propto \frac{\rho U^2 L^2}{\sigma L} = \frac{\rho U^2 L}{\sigma}
\]

- Similarity requirement for model and full scale forces:

\[
\left(\frac{\rho U^2 L}{\sigma}\right)_M = \left(\frac{\rho U^2 L}{\sigma}\right)_F \quad \sigma = 0.073 \text{ at } 20^\circ\text{C}
\]

\[
\frac{U_M}{\sqrt{\sigma_M/\rho L_M}} = \frac{U_F}{\sqrt{\sigma_F/\rho L_F}} = W_n
\]

When $W_n > 180$, we assume that a further increase in $W_n$ doesn’t influence the fluid forces.
Scaling ratios used in testing of ships and offshore structures

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Dimensionless Number</th>
<th>Force Ratio</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_e$</td>
<td>Reynolds Number</td>
<td>Inertia/Viscous</td>
<td>$\frac{UL}{\nu}$</td>
</tr>
<tr>
<td>$F_n$</td>
<td>Froude Number</td>
<td>Inertia/Gravity</td>
<td>$\frac{U}{\sqrt{gL}}$</td>
</tr>
<tr>
<td>$M_n$</td>
<td>Mach’s Number</td>
<td>Inertia/Elasticity</td>
<td>$\frac{U}{\sqrt{E_v / \rho}}$</td>
</tr>
<tr>
<td>$W_n$</td>
<td>Weber’s Number</td>
<td>Inertia/Surface tension</td>
<td>$\frac{U}{\sqrt{\sigma / \rho L}}$</td>
</tr>
<tr>
<td>$St$</td>
<td>Strouhall number</td>
<td>-</td>
<td>$\frac{f_v D}{U}$</td>
</tr>
<tr>
<td>$KC$</td>
<td>Keulegan-Carpenter Number</td>
<td>Drag/Inertia</td>
<td>$\frac{U A T}{D}$</td>
</tr>
</tbody>
</table>
Froude Scaling

\[ \frac{U_M}{\sqrt{gL_M}} = \frac{U_F}{\sqrt{gL_F}} \implies U_F = U_M \sqrt{\frac{L_F}{L_M}} = U_M \sqrt{\lambda} \]

Using the geometrical similarity requirement: \( \lambda = \frac{L_F}{L_M} \)

If you remember this, most of the other scaling relations can be easily derived just from the physical units.
## Froude scaling table

<table>
<thead>
<tr>
<th>Physical Parameter</th>
<th>Unit</th>
<th>Multiplication factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>[m]</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Structural mass:</td>
<td>[kg]</td>
<td>$\lambda^3 \cdot \rho_F / \rho_M$</td>
</tr>
<tr>
<td>Force:</td>
<td>[N]</td>
<td>$\lambda^3 \cdot \rho_F / \rho_M$</td>
</tr>
<tr>
<td>Moment:</td>
<td>[Nm]</td>
<td>$\lambda^4 \cdot \rho_F / \rho_M$</td>
</tr>
<tr>
<td>Acceleration:</td>
<td>[m/s²]</td>
<td>$a_F = a_M$</td>
</tr>
<tr>
<td>Time:</td>
<td>[s]</td>
<td>$\sqrt{\lambda}$</td>
</tr>
<tr>
<td>Pressure:</td>
<td>[Pa=N/m²]</td>
<td>$\lambda \cdot \rho_F / \rho_M$</td>
</tr>
</tbody>
</table>
Hydroelasticity

• Additional requirements to the elastic model
  – Correctly scaled global stiffness
  – Structural damping must be similar to full scale
  – The mass distribution must be similar

• Typical applications:
  – Springing and whipping of ships
  – Dynamic behaviour of marine risers and mooring lines
Scaling of elasticity

Hydrodynamic force:

Geometric similarity requirement:

Requirement to structural rigidity:
Scaling of elasticity
– geometrically similar models

• Geometrically similar model implies: \( I_F = I_M \lambda^4 \)

• Must change the elasticity of material: \( E_F = E_M \lambda \)

• Elastic propellers must be made geometrically similar, using a very soft material: \( E_M = E_F / \lambda \)

• Elastic hull models are made geometrically similar only on the outside. Thus, E is not scaled and \( I_M = I_F \cdot \lambda^{-5} \)
Cavitation

- Dynamic similarity requires that cavitation is modelled
- Cavitation is correctly modelled by equality in cavitation number:

\[ \sigma = \frac{(\rho gh + p_0) - p_v}{1/2 \rho U^2} \]

- To obtain equality in cavitation number, atmospheric pressure \( p_0 \) might be scaled
- \( p_v \) is vapour pressure and \( \rho gh \) is hydrostatic pressure
- Different ”definitions” of the velocity \( U \) is used
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