TMR4205 Buckling and Ultimate Strength of Marine Structures

Chapter 5:
Buckling of Cylindrical Shells

by
Professor Jørgen Amdahl

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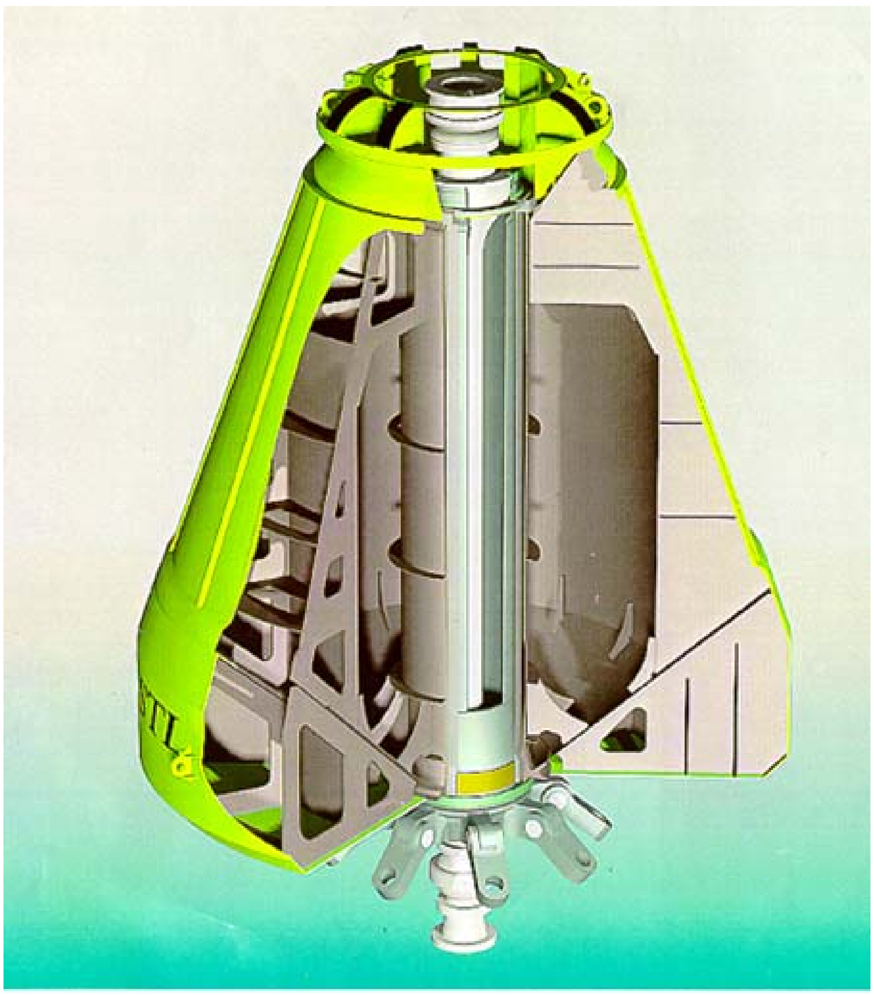


CONTENTS

5. BUCKLING OF CYLINDRICAL SHELLS

5.1 Introduction

Stiffened and unstiffened cylindrical shells are important structural elements in offshore structures. They are very often subjected to compressive stresses and must be designed against buckling criteria. The buckling behaviour is usually more violent than it is for plate and column structures.

A theoretical load-end shortening curve representative for cylindrical shells subjected to axial compression is shown in Figure 5.1. During initial loading the structure follows the linear primary equilibrium path. At some load level, the primary path is intersected by an unstable secondary path. The buckling mode in the secondary path is quite different from the deformations in its stable primary state of equilibrium. The intersection is called a *bifurcation* point, B.

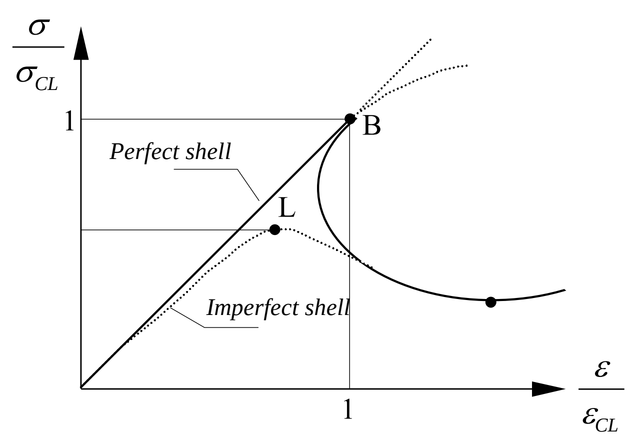
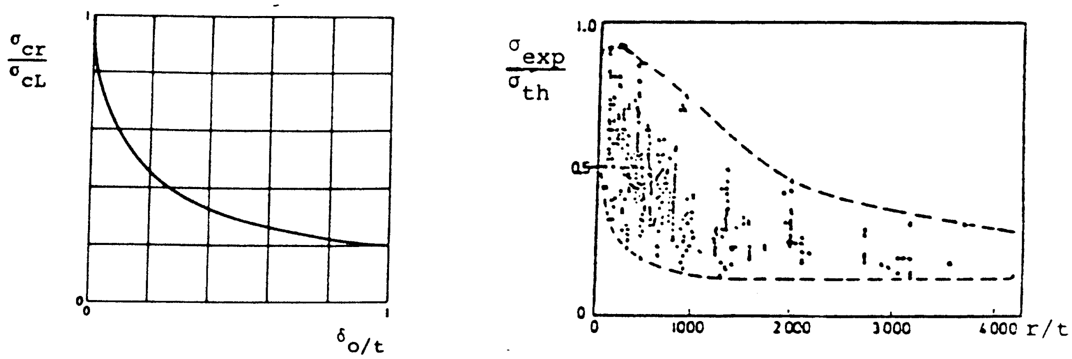


Figure 5.1 Equilibrium Paths for Perfect and Imperfect Shells.

In practice, it is very difficult to reach the theoretical bifurcation loads. The reason for this is the presence of initial imperfections which causes the shell to fail into a configuration close to the buckled shape of the ideal cylinder at a load significantly smaller than the bifurcation load. Therefore, buckling occurs at the limit point L rather than the bifurcation load B.

Figure 5.2a shows the enormous influence of a small axisymmetric initial imperfection, δ_0, on the buckling load, N, of an axially loaded cylinder. For an imperfection amplitude of only 1/10 of the wall thickness, the buckling load is reduced to 60% of the theoretical value, N_{cr}. The imperfection sensitivity is further illustrated in the plot of experimental buckling loads in Figure 5.2b, where the wide scatter of the results is also observed. Because of this effect, the design of cylindrical shells is based on the modification of the theoretical load by an empirical reduction, or a *knock-down* factor.



(a) (b)

Figure 5.2 (a) Influence of Axisymmetric Imperfections on the Buckling Load of a Cylinder, (b) Experimental Buckling Loads Of Axially Loaded Cylinders.

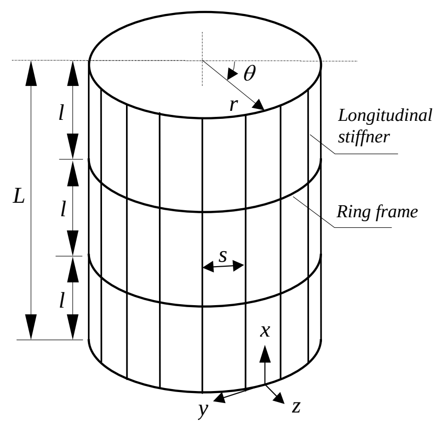


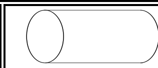
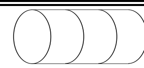
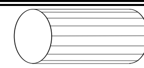
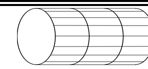
Figure 5.3 Geometrical Parameters of a Stiffened Cylindrical Shell, /5.1/.

The characteristic geometric parameters of a stiffened cylindrical shell is defined in Figure 5.3. The buckling modes for stiffened cylindrical shell may be categorized as follows:-

- *Shell buckling*;
  Buckling of shell plating between stiffeners and frames.
- *Interframe shell buckling*;
  Involves buckling of the longitudinal stiffener with associated shell plating.
- *Panel ring buckling*;
  Buckling of rings with associated plate flange between longitudinal stiffeners.
- *General buckling*;
  Involves bending of shell plating, longitudinal stiffeners as well as ring frames.
- *Torsional or local buckling of stiffeners and frames*.
- *Column buckling of the cylinder*.

Possible buckling modes for cylinders with various stiffener arrangements are displayed in Table 5-1.

Table 5-1 Buckling Modes in Cylinders with Various Stiffener Arrangements, /5.2/.

Buckling Mode	Geometry				
SHELL BUCKLING	Unstiffened cylinder	X	X		
	Unstiffened curved panel			X	X
PANEL BUCKLING	Stringer stiffened cylinder			X	X
GENERAL BUCKLING	Ring stiffened cylinder		X		
	Ring / Stringer stiffened cylinder				X
OVERALL BUCKLING	Column	X	X	X	X
LOCAL STIFFENER BUCKLING	Ring		X		X
	Stringer			X	X

5.2 Equilibrium Equations for Cylindrical Shells

The classical theory for buckling of cylindrical shells, as suggested by Flügge, may be found in /5.3/. Analytic solutions to the Flügge equations are known for several load cases. However, for practical purposes it is more convenient to use the equations proposed by Donnel /5.4/ which leads to very simple formulas for the buckling stress. The simplifications introduced somewhat limit to their range of applicability, but it has been found that they may be used under the following restrictions, /5.5/.

- The number of circumferential waves should not be too small, ($n \sqsupseteq 4$)

Stress state for a cylindrical shell.

- The deformations in the longitudinal and circumferential direction are small as compared to the radial displacement. Therefore, the column buckling mode can not be predicted.

Figure 5.4 shows an infinitesimal element of a shell with its associated stress resultants from membrane and bending actions. Considering equilibrium in the axial, circumferential, and radial directions, the following equations are obtained,

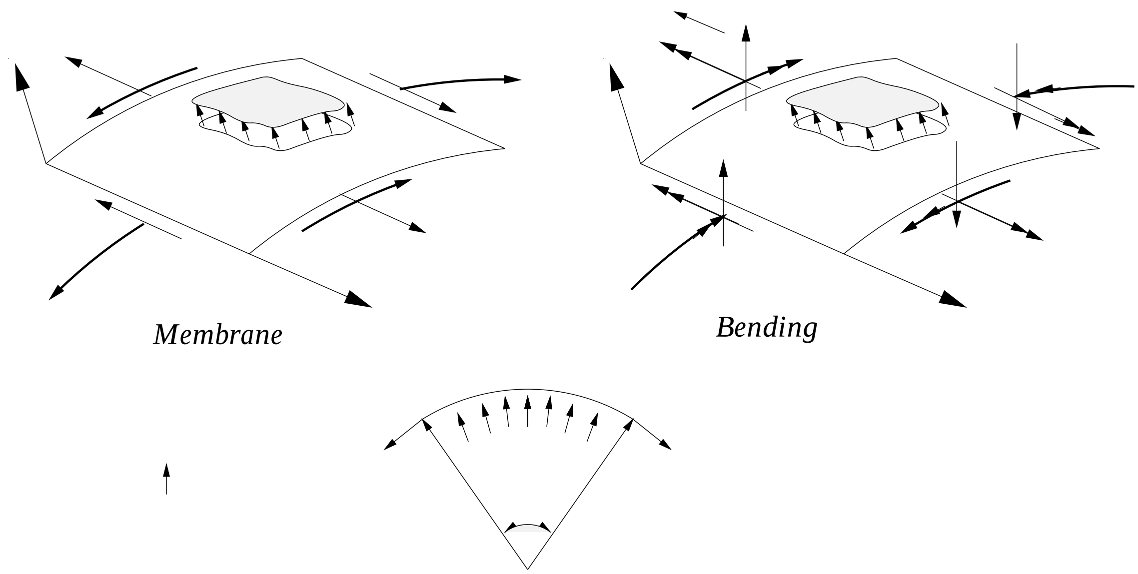


Figure 5.4 Shell Stress Resultants.

$$r\frac{\partial N_x}{\partial x}+\frac{\partial N_{\theta x}}{\partial \theta}=0 \tag{5.1}$$

$$r\frac{\partial N_{x\theta}}{\partial x}+\frac{\partial N_{\theta}}{\partial \theta}=0 \tag{5.2}$$

$$\nabla^4 w=\frac{1}{D}\left(p+N_x\frac{\partial^2 w}{\partial x^2}+\frac{2}{r}N_{x\theta}\frac{\partial^2 w}{\partial x\partial\theta}+\frac{1}{r^2}N_\theta\frac{\partial^2 w}{\partial\theta^2}-\frac{1}{r}N_\theta\right) \tag{5.3}$$

where,

$$\begin{aligned} N_x &= \sigma_x t \\ N_{x\theta} &= N_{\theta x} = \sigma_{x\theta} t \\ N_\theta &= \sigma_\theta t \end{aligned} \tag{5.4}$$

$$\nabla^4=\left(\nabla^2\right)^2=\left(\frac{\partial^2}{\partial x^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \tag{5.5}$$

The plate stiffness, D, is given by

$$D=\frac{Et^3}{12\left(1-\nu^2\right)} \tag{5.6}$$

The pressure, p, is positive outwards. Note the similarity between Equation (5.3) and the corresponding expression for plate equilibrium. The equations can simply be obtained from the plane membrane and plate equations by substituting

$$\frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial \theta} \quad , \quad \frac{\partial^2}{\partial y^2} = \frac{1}{r^2}\frac{\partial^2}{\partial^2 \theta} \tag{5.7}$$

The only new term is $1\,N_\theta/r$ which represents the lateral component of the circumferential stress. Thus, unlike plates, cylindrical shells can carry lateral loads by pure membrane action and no bending. This is a very efficient property, but at the same time this makes shells sensitive to buckling.

Equations (5.1-3) form a coupled set of three non-linear equations with four variables-- N_x, $N_{x\theta}$, N_θ, and w. The circumferential stress, N_θ, due to *uniform* lateral deformation is related to w by

$$\varepsilon_x = \frac{w}{r} \qquad N_x = E\varepsilon_x t = Et\frac{w}{r} \tag{5.8}$$

Introducing this expression and applying the operator ∇, Equation (5.3) may also be written as,

$$\nabla^8 w = \frac{\nabla^4}{D}\left(N_x\frac{\partial^2 w}{\partial x^2} + \frac{2}{r}N_{x\theta}\frac{\partial^2 w}{\partial x\partial\theta} + \frac{1}{r^2}N_\theta\frac{\partial^2 w}{\partial\theta^2}\right) - \frac{Et}{Dr^2}\frac{\partial^4 w}{\partial x^4} \tag{5.9}$$

This expression is denoted Donnel's equation and constitutes the basis for the derivation of the elastic buckling stresses. By contrast to buckling of stiffened plates, classical buckling theory is crucial in determination of the critical stresses.

5.3 Stress Analysis

5.3.1 Beam Theory

A cylindrical shell is generally exposed to the load conditions shown in Figure 5.5.

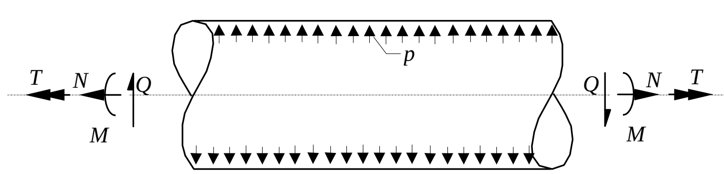


Figure 5.5 Load Conditions in a Cylindrical Shell.

The stress can often be derived by assuming that the wall thickness is much smaller than the radius. The axial stress becomes

$$\sigma_x = \sigma_a + \sigma_b \tag{5.10}$$

where the mean axial stress is

$$\sigma_a = \frac{N}{2\pi rt} \tag{5.11}$$

and the bending stress is,

$$\sigma_b = \frac{Mz}{I} = \frac{Mr\sin\theta}{\pi r^3 t} = \frac{M\sin\theta}{\pi r^2 t} \tag{5.12}$$

where θ denotes the angular position of the stress point relative to the *y*-axis. If the shell is provided with longitudinal stiffeners, these can be smeared so as to obtain an equivalent thickness,

$$t_e = t + \frac{A}{s} \tag{5.13}$$

where *A* is the area of stiffener without plate flange and *s* is the stiffener spacing.

The total shear stress is given by

$$\tau = \tau_T + \tau_Q \tag{5.14}$$

where the expressions for the shear stress due to torsion and the shear stress due to bending are given, respectively, as

$$\tau_T = \frac{T}{2A_{cyl}t} = \frac{T}{2\pi r^2 t} \tag{5.15}$$

and

$$\tau_Q = \frac{QS}{It} = \frac{Qr^2 t\cos\theta}{\pi r^3 t \cdot t} = \frac{Q}{\pi \cdot rt}\cos\theta \tag{5.16}$$

Stiffeners are normally not considered to influence the shear stresses.

5.3.2 Lateral Pressure

For an unstiffened cylinder subjected to lateral pressure, the circumferential stress can be obtained by considering equilibrium of a half section as shown in Figure 5.6.

$$\begin{aligned} &2\sigma_\theta \cdot t = p \cdot 2r \\ &\Rightarrow \sigma_\theta = \frac{pr}{t} \qquad (\text{Tension if } \sigma_\theta > 0) \end{aligned} \tag{5.17}$$

For a closed cylinder the axial stress becomes

$$2\pi rt\sigma_x = \pi r^2 p$$
$$\Rightarrow \sigma_x = \frac{pr}{2t} = \frac{1}{2}\sigma_\theta \tag{5.18}$$

The circumferential stress may also be determined directly from the equilibrium Equation (5.3) because the entire load is carried by the membrane action (*w* is constant around the perimeter so that all the derivates vanish).

$$p - \frac{1}{r}N_\theta = 0$$
$$\Rightarrow \sigma_\theta = \frac{pr}{t} \tag{5.19}$$

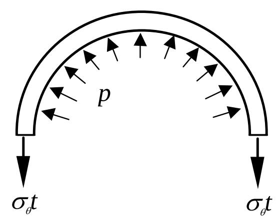


Figure 5.6 Half Section of Cylinder with Lateral Pressure.

The circumferential stresses in a ring stiffened cylinder subjected to lateral pressure and axial stress may be derived by a simple consideration. Assume that the ring has an associated plate flange with effective width, l_{eo}, so that the effective thickness in the ring direction is

$$t_e = t + \frac{A_r}{\ell_{eo}} \tag{5.20}$$

also, assume first that the ring is infinitely rigid. Due to restrained transverse contraction the axial stress, σ_x, causes a circumferential stress, $\nu\sigma_x$, (tension if $\sigma_x > 0$). The net <u>external</u> force acting on each half section is accordingly

$$pr - \nu\sigma_x t \tag{5.21}$$

When the ring is allowed to deform, a circumferential stress, σ_θ^r, 2is set up in the ring and plate flange. Equilibrium yields

$$\sigma_\theta^r t_e = pr - \nu\sigma_x t \tag{5.22}$$

or,

$$\sigma_\theta^r = \left(\frac{pr}{t} - \nu\sigma_x\right)\frac{1}{1+\frac{A_r}{\ell_{eo}t}} \tag{5.23}$$

In the plate flange at the ring the stress due to restrained contraction must be added, so that

$$\sigma_\theta^{plate} = \left(\frac{pr}{t} - \nu\sigma_x\right)\frac{1}{1+\dfrac{A_r}{\ell_{eo}t}} + \nu\sigma_x \tag{5.24}$$

If σ_x is due to the end pressure alone, and p is constant (σ_x = $pr/2t$), the formula can be written as

$$\sigma_\theta^{plate} = \frac{pr}{t}\left(\frac{1-\dfrac{\nu}{2}}{1+\dfrac{A_r}{\ell_{eo}t}} + \frac{\nu}{2}\right) \tag{5.25}$$

The circumferential stress **mid-way** between the rings depends on the size of the stiffeners and the distance between the rings. For an unstiffened cylinder, the circumferential stress is given by Equation (5.16). For a cylinder with closely spaced ring stiffeners, the stress is given by Equation (5.21). It is natural to assume that the circumferential stress, in the general case, can be obtained by interpolation between these two extreme cases so that,

$$\sigma_\theta = \frac{pr}{t}(1-\xi) + \xi\left(\left(\frac{pr}{t} - \nu\sigma_x\right)\frac{1}{1+\dfrac{A_r}{\ell_{eo}t}} + \nu\sigma_x\right) \tag{5.26}$$

where,

$$\begin{cases} \xi \to 0 & \text{for very distant stiffners} \\ \xi \to 1 & \text{for closely spaced stiffners} \end{cases}$$

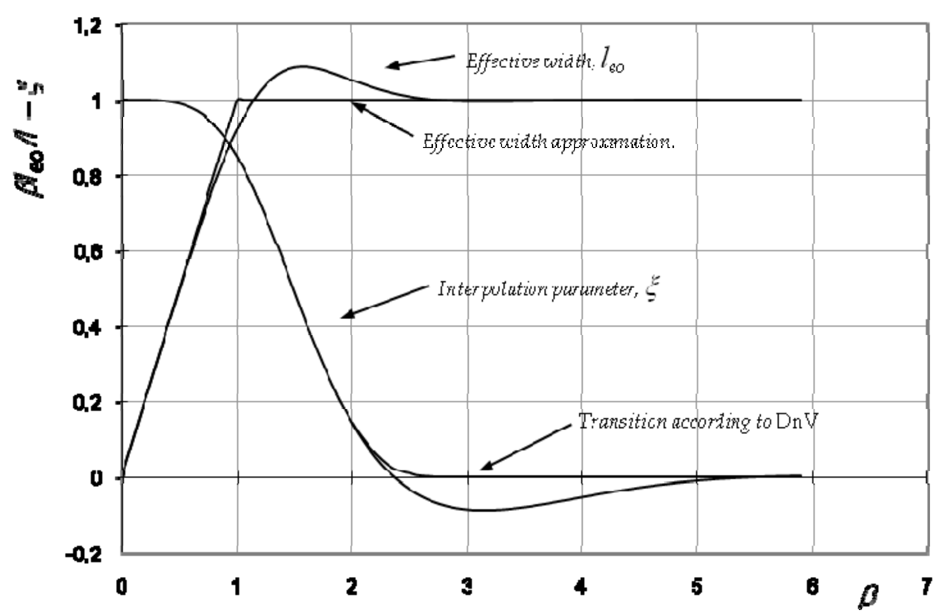


Figure 5.7 The Parameters l_{eo} and ζ.

Rearranging this gives

$$\sigma_\theta = \frac{pr}{t} - \frac{\frac{A_r}{\ell_{eo}t}}{1+\frac{A_r}{\ell_{eo}t}}\xi\left(\frac{pr}{t} - \nu\sigma_x\right) \tag{5.27}$$

This formula can be derived from the solution to the differential equation of equilibrium as shown below. The parameters are defined as follows,

$$\zeta = 2\frac{\sinh\beta\cos\beta + \cosh\beta\sin\beta}{\sinh 2\beta + \sin 2\beta} \tag{5.28}$$

$$\beta = \frac{\ell}{1.56\sqrt{rt}} \quad \approx \frac{2}{3}\sqrt{Z} \tag{5.29}$$

where Z is the so called Batdorf' parameter. The effective width is given by,

$$\ell_{eo} = \frac{\ell}{\beta}\frac{\cosh 2\beta - \cos 2\beta}{\sinh 2\beta + \sin 2\beta} \tag{5.30}$$

The parameters ζ and l_{eo} are plotted in Figure 5.7. It is seen that $l_{eo} \approx l$ for closely spaced stiffeners and approaches asymptotically $1.56(rt)^{0.5}3$ for long cylinders. A good approximation, which is also plotted in Figure 5.7, is obtained as,

$$\ell_{eo} = \min\left\{\ell, 1.56\sqrt{rt}\right\} \tag{5.31}$$

5.3.3 Lateral Pressure-- Solution of the Differential Equation

(*A Cylinder under Lateral Pressure and Axial Stress*).

The deformation will in this case be axisymmetric so the differential equation reads,

$$D\,w_{,xxxx} + \frac{1}{r}N_\theta - N_x w_{,xx} = p \tag{5.32}$$

For axisymmetric loading the circumferential strain is given by

$$\varepsilon_\theta = \frac{w}{r} \tag{5.33}$$

and

$$N_\theta = E\varepsilon_\theta \cdot t + \nu N_x \quad = Et\frac{w}{r} + \nu t\sigma_x \tag{5.34}$$

This gives,

$$D\,w_{,xxxx} - \sigma_x t\,w_{,xx} + \frac{Et}{r^2}w = p - \frac{\nu t\sigma_x}{r} = p_e \tag{5.35}$$

where p_e is the effective load. In the following linear theory is applied, i.e. the large deflection term $\sigma_x tw_{,xx}4$ is neglected. By introducing the parameter,

$$4k^4 = \frac{Et}{Dr^2} = \frac{12(1-\nu^2)}{r^2t^2} \tag{5.36}$$

the equation takes the form,

$$w_{,xxxx} + 4k^4 w = \frac{1}{D}p' \tag{5.37}$$

The solution to this differential equation is given as,

$$w = w_p + w_h \tag{5.38}$$

where the particular and the homogeneous solutions can be written, respectively, as

$$w_p = \frac{p'}{4Dk^4} = \left(\frac{pr}{t} - \nu\sigma_x\right)\frac{r}{Et} \tag{5.39}$$

and,

$$w_h = e^{\beta x}(C_1 \cos kx + C_2 \sin kx) + e^{-\beta x}(C_3 \cos kx + C_4 \sin k) \tag{5.40}$$

The integration constants are determined form the boundary conditions. In the following, the case illustrated in Figure 5.8 is studied. The coordinate system is located midway between the rings

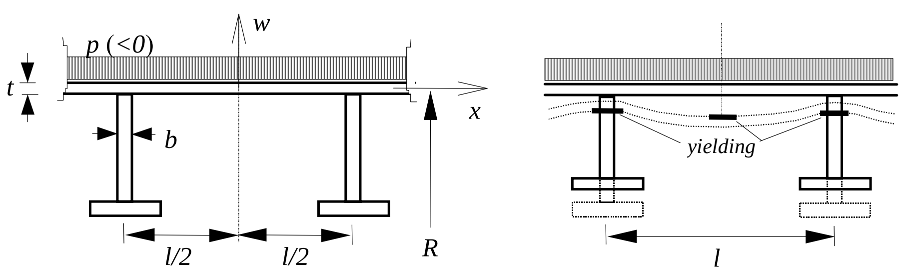


Figure 5.8 Ring Stiffened Cylinder Exposed to Lateral Pressure.

Symmetry yields $w(-x) = w(x)$ which implies,

$$C_1 = C_3, \quad C_2 = -C_4 \tag{5.41}$$

$$\begin{aligned} w &= w_p + C_1 \cos kx\left(e^{kx} - e^{-kx}\right) + C_2 \sin kx\left(e^{kx} - e^{-kx}\right) \\ &= w_p + A_1 \cos kx \cosh kx + A_2 \sin kx \sinh kx \end{aligned} \tag{5.42}$$

Also $w_{,x} = 0$ for $x = l/2$. This yields,

$$w_{,x} = A_1 k \sin\beta\cosh\beta + A_1 k\cos\beta\sinh\beta + A_x k\cos\beta\sinh\beta + A_2 k\sin\beta\cosh\beta = 0 \tag{5.43}$$

where,

$$\beta = \frac{k\ell}{2} \quad \Rightarrow \beta^2 = \frac{k^2\ell^2}{4} = \frac{\sqrt{3}}{4}\frac{\sqrt{(1-\nu^2)}\ell^2}{rt} = \frac{\sqrt{3}Z}{4} \tag{5.44}$$

This gives

$$A_2 = A_1\frac{\sin\beta\cosh\beta - \cos\beta\sinh\beta}{\sin\beta\cosh\beta + \cos\beta\sinh\beta} \tag{5.45}$$

The condition that the displacement should be equal to the ring frame displacement at the ring yields,

$$\begin{aligned} w_R - w_p &= A_1\cos\beta\cosh\beta + A_1\frac{\sin\beta\cosh\beta - \cos\beta\sinh\beta}{\sin\beta\cosh\beta + \cos\beta\sinh\beta}\sin\beta\sinh\beta \\ &= A_1\frac{1}{2}\frac{\sin 2\beta + \sinh 2\beta}{\sin\beta\cosh\beta + \cos\beta\sinh\beta} \end{aligned} \tag{5.46}$$

which implies that ,

$$\begin{aligned} A_1 &= 2\left(w_R - w_p\right)\frac{\sinh\beta\cos\beta + \cosh\beta\sin\beta}{\sinh 2\beta + \sin 2\beta} \\ &= \left(w_R - w_p\right)\xi \end{aligned} \tag{5.47}$$

when Equation (5.27) is employed.

Observing that the term related to A_2 vanishes for $x = 0$, the deflection at mid-way between the rings is,

$$\begin{aligned} w_{x=0} &= w_p + \left(w_R - w_p\right)\xi \\ &= w_p\left(1-\xi\right) + w_R\xi \end{aligned} \tag{5.48}$$

Thus, the first term represents the contribution from an unstiffened cylinder, and the second term the contribution from a cylinder with closely spaced stiffeners as shown by the physical reasoning in Section 5.3.2.

The circumferential stress due to w_p is obtained from Equation (5.38),

$$\sigma_\theta^p = E\frac{w_p}{r} + \nu\sigma_x = \frac{pr}{t} \tag{5.49}$$

The total force in the circumferential direction between two rings is given by,

$$
\begin{aligned}
F_\phi &= 2\int_0^{l/2} \sigma_\theta t\, dx \quad = 2t\int_0^{l/2} E\frac{w}{r}dx \\
&= 2t\frac{E}{r}\left[w_p \cdot \frac{l}{2} + A_1 \int_0^{l/2} \left(\cosh kx \cos kx\right) dx + A_2 \int_0^{l/2} \left(\sinh kx \sin kx\right) dx \right] \qquad (5.50) \\
&= \frac{Et}{r}\left[w_p l + A_1 \frac{1}{k}\left(\sinh\beta\cos\beta + \cosh\beta\sin\beta\right) + A_2\frac{1}{k}\left(\cosh\beta\sin\beta - \sinh\beta\cos\beta\right)\right]
\end{aligned}
$$

Introducing the expressions for A_1 and A_2 there is obtained

$$
F_\phi = \frac{Et}{r}\left[w_p l + \left(w_R - w_p\right)\frac{2}{k}\frac{\cosh 2\beta - \cos 2\beta}{\sinh 2\beta + \sin 2\beta}\right] \qquad (5.51)
$$

The last term has a length dimension, and is interpreted as an effective length denoted by

$$
l_{eo} = \frac{2}{k}\frac{\cosh 2\beta - \cos 2\beta}{\sinh 2\beta + \sin 2\beta} = \frac{l}{\beta}\frac{\cosh 2\beta - \cos 2\beta}{\sinh 2\beta + \sin 2\beta} \qquad (5.52)
$$

Therefore, Equation (5.50) becomes,

$$
F_\phi = \frac{Et}{r}\left[w_p l + \left(w_R - w_p\right) l_{eo}\right] \; = \frac{Et}{r}\left[w_R l_{eo} + w_p\left(l - l_{eo}\right)\right] \qquad (5.53)
$$

Hence, the total force in the circumferential direction can be considered to consist two blocks:__ one with the stress of the ring stiffener acting over the effective flange, l_{eo}, and the remainder, $l - l_{eo}$, has a stress equal to an unstiffened cylinder.

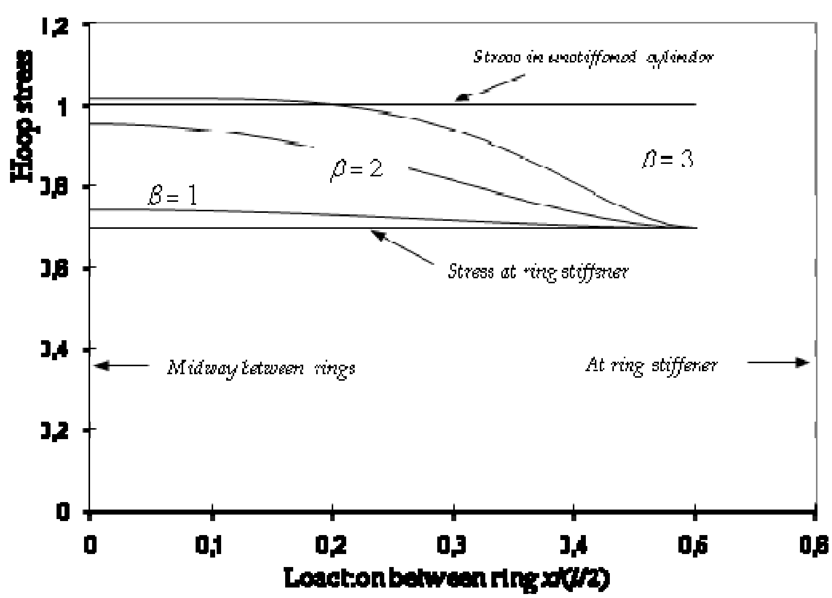


Figure 5.9 Stress Distribution Over a Half-Ring Frame.

The exact stress distribution can be obtained by introducing the displacement function in the stress expression. In Figure 5.9 the stress distribution over a half-ring frame is sketched for three different β-values. For β=1 the rings are closely spaced and the stress mid-way between the rings is almost equal to the stress at the ring stiffener. For β=3 the stress is approximately equal to the stress in an unstiffened cylinder for about 60 % of the frame spacing. Only in the vicinity of the rings is their effect noticeable. The effective flange for β = 1, 2, and 3, is $l_{eo}/l \approx$ 0.92, 0.54, and 0.34, respectively.

5.4 Buckling of Cylinders

5.4.1 Axial Compression

Consider a cylinder subjected to an axial compressive load, P. If the end effects are neglected, the following assumptions apply,

$$N_x = \frac{P}{2\pi r} \quad , \quad N_{x\theta} = N_\theta = 0 \tag{5.54}$$

Introduction of these values into Equation (5.8) gives

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} + \frac{P}{2\pi r}\nabla^4\left(\frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{5.55}$$

The solution to this differential equation takes the form

$$w = \delta\left(\sin\frac{m\pi x}{l}\right)\sin n\theta\left(\frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{5.56}$$

where m is the number of half waves in the longitudinal direction and n is the number of entire waves in circumferential direction which gives for the critical stress

$$\sigma_{xE} = \frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{l}\right)^2\left[\frac{\left(m^2+\bar{n}^2\right)^2}{m^2} + \frac{12Z^2}{\pi^4}\frac{m^2}{\left(m^2+\bar{n}^2\right)^2}\right] \tag{5.57}$$

where Z is the Batdorf parameter,

$$Z = \frac{l^2}{rt}\sqrt{\left(1-\nu^2\right)} \tag{5.58}$$

and

$$\bar{n} = \frac{nl}{\pi r} \tag{5.59}$$

For cylinders of intermediate length, a close estimate of the smallest critical load may be obtained by analytical minimization of Equation (5.56) with respect to the quantity

$$\left(\frac{m^2+\bar{n}^2}{m}\right)^2 \tag{5.60}$$

Then, the <u>minimum</u> is found for

$$\left(\frac{m^2+\bar{n}^2}{m}\right)^2=\frac{2\sqrt{3}}{\pi^2}Z \tag{5.61}$$

which gives the following critical load,

$$\sigma_{xE}=\frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{l}\right)^2\cdot\frac{4\sqrt{3}}{\pi^2}Z=0.605\frac{Et}{r}=\sigma_{cl} \tag{5.62}$$

This is the classical solution for axially compressed cylinder. The term,

$$C_Z^{(2)}=\frac{4\sqrt{3}}{\pi^2}Z \tag{5.63}$$

is interpreted as the buckling coefficient for an intermediate length cylinder. It is observed from Equation (5.60) that several buckling modes may correspond to a single bifurcation point as illustrated in Figure5.10. It should be noted that m and $\bar{n}$ are treated as continuous variables in the minimization process while they are in effect discrete quantities. The error introduced by this is, however, minimal as seen from Figure5.10 where the coefficient is plotted as a function of m and n.

It is also observed that the buckling stress is only dependent on the r/t ratio and independent of the length l . To illustrate, r/t = 605 gives a σ_{xE} = 210 MPa. However, as discussed in Section 5.1, the elastic buckling stress obtained for real structures could be in the range of 40% for this r/t ratio. Hence, in order to obtain this value r/t ~ 250 will be required. Obviously, cylinders with shell thickness of 20 mm and diameter in the range of 10 m need to be stiffened.

For short cylinders the buckling mode will be axisymmetric with m=1 and n=0. The following buckling coefficient,

$$C_Z^{(1)}=1+\frac{12Z^2}{\pi^4} \tag{5.64}$$

is valid for

$$Z<Z^{(1)}=\frac{\pi^2}{2\sqrt{3}}=2.85 \tag{5.65}$$

An approximate buckling coefficient which is valid for short as well as long shells, may be obtained by applying the elliptical interaction formula,

$$\left(\frac{1}{C_Z}\right)^2 + \left(\frac{C_Z^{(2)}}{C_Z}\right)^2 = 1 \qquad (5.66)$$

It is seen that C_Z approaches asymptotically the correct values for small and large values of the Batdorf parameter, *Z*. Rearranging Equation (5.65) there comes out

$$C_Z = \sqrt{1 + \frac{48Z^2}{\pi^4}} \qquad (5.67)$$

The first term can be interpreted as pertaining to buckling of a plane wide plate, the second is the *shell* contribution.

For long cylinders, column buckling is a potential collapse mode. The buckling stress for a shallow shell is expressed by

$$\sigma_E = \frac{\pi^2 EI}{Al^2} \approx \frac{\pi^2 E}{2}\left(\frac{r}{l}\right)^2 \qquad (5.68)$$

Thus, the Donnel's theory is valid for

$$Z < Z^{(3)} = \frac{\pi^2\sqrt{3}}{2}\left(1-\nu^2\right)\left(\frac{r}{t}\right)^2 \qquad (5.69)$$

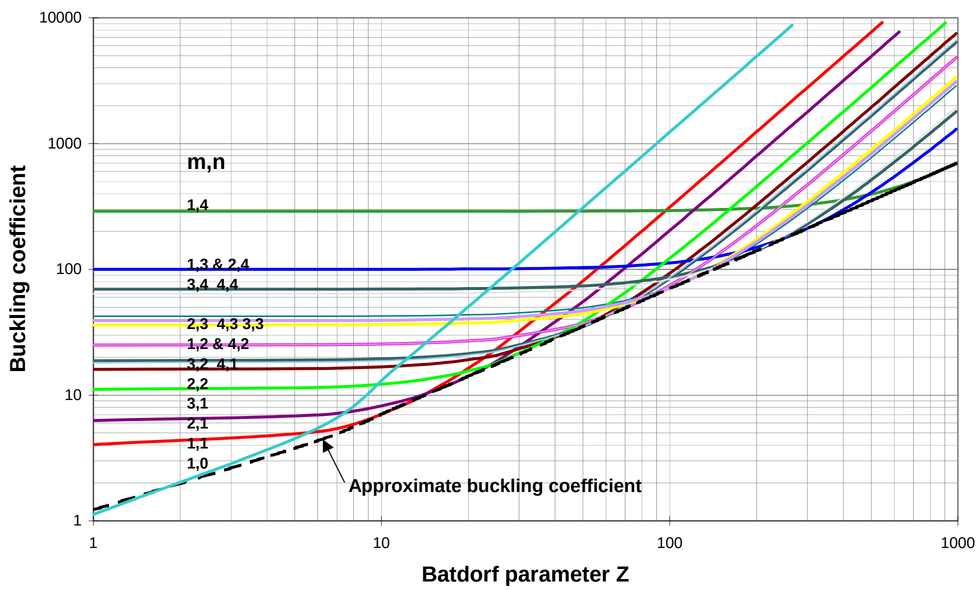


Figure5.10 The buckling coefficient for various buckling modes {m,n}

5.4.2 Curved Panel

The buckling load for a curved panel of width, *s*, and length, *l*, can be found by introducing the term

$$n = \frac{\pi r k}{s} \tag{5.70}$$

into Equation (5.55), where *k* is the number of half waves across the width. Obviously, for widely stiffened cylinders with large s, Equation (5.61) applies and the buckling stress is independent of the stiffener spacing.

By reducing the stiffener spacing, the panel becomes narrow. By rearranging Equation (5.56), the following expression emerges for narrow panels ($k = 1$).

$$\sigma_E = \frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{s}\right)^2\left(4+\frac{3Z_S^2}{\pi^4}\right) \tag{5.71}$$

where the Batdorf parameter now reads,

$$Z_S = \frac{s^2}{rt}\sqrt{1-\nu^2} \tag{5.72}$$

The last bracket in Equation (5.70) is interpreted as the buckling coefficient, $\bar{C}_Z^{(4)}$, which is valid for

$$Z_S < Z_S^{(2)} = \frac{2\pi^2}{\sqrt{3}} = 11.4 \tag{5.73}$$

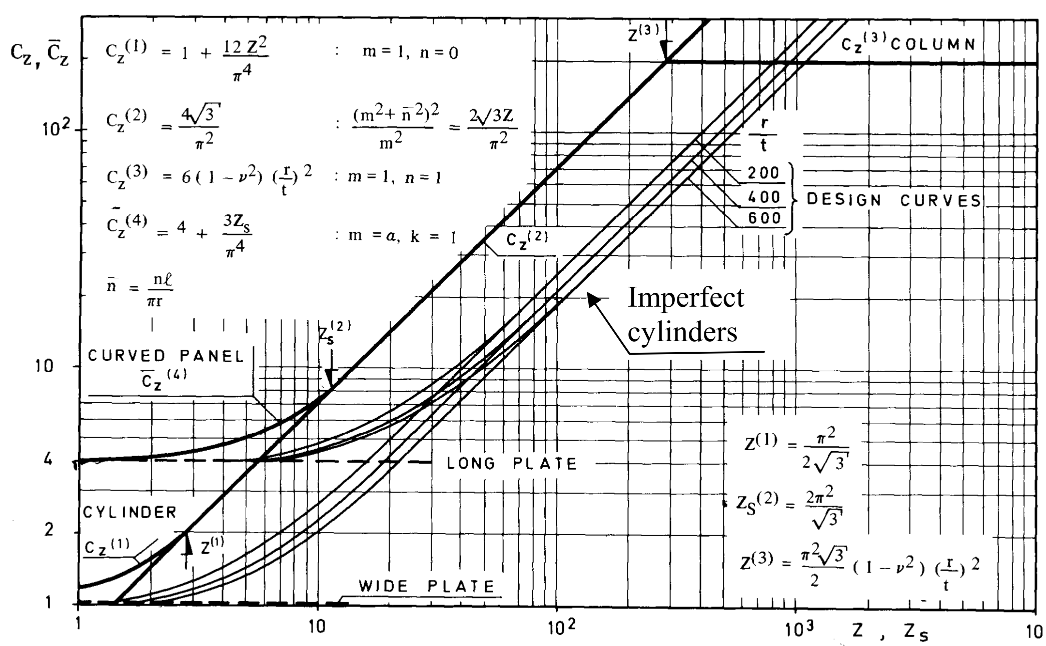


Figure 5.11 Buckling Coefficients For Axial Compression.

An approximate buckling coefficient valid for the entire interval of Z_S may be obtained in a manner similar to that for closed cylinders, (see Equation 5.65)

$$\overline{C}_Z = 4\sqrt{1 + \frac{3Z_S^2}{\pi^4}} \tag{5.74}$$

Again, the first term can be considered pertaining to buckling of a plane, long plate, the second term pertains to the curved *shell*.

The various buckling coefficients are shown as a function of the Batdorf parameter in Figure 5.11.

5.4.3 Bending

It is considerably more complicated to analyze cylinders subjected to bending because,

- The initial stress distribution is no longer constant around the circumference.
- The pre-buckling deformations of long cylinders is highly non-linear due to ovalization of the cross-section.

However, studies carried out in this field indicates that the buckling resistance due to bending may be taken equal to the buckling stress for axial compression for all practical purposes.

5.4.4 The effect of stiffening

Figure 5.11 illustrates the effect of applying longitudinal and ring stiffeners with respect to buckling for axial compression. The coefficient $C_z^{(2)}$ represents the solution where the stiffener spacing is too large to have any influence on the buckling resistance. For longitudinal stiffeners $Z_s^{(2)} = 11.4$ indicates the transition where the panel becomes narrow and the stiffeners start to influence the buckling resistance. This applies to perfect cylinders. When effect of imperfection is taken into account the transition increases, because the contribution from plate buckling (which is independent of imperfections) increases compared to the contribution from shell buckling. In practice the effect of stiffeners becomes noticeable for $Z_s < \sim 50$. This corresponds to $\frac{s}{t} \leq 7\sqrt{\frac{r}{t}}$ $(l>s)$. To illustrate, for a column with diameter of 10 m, shell thickness 20 mm, the maximum spacing for the stiffeners to improve the buckling resistance is approximately 2.2 m. Of course, longitudinal stiffeners contribute in any case to the utilization by reducing the axial stress.'

5.4.5 External Lateral Pressure

In the pre-buckling state the external pressure sets up compressive membrane stresses in the meridional direction. Retaining only the linear terms in Equation (5.3) gives,

$$N_\theta = -pr \tag{5.75}$$

Introduction into Equation (5.8) yields the following stability equation

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} + \frac{1}{r}p\nabla^4\left(\frac{\partial^2 w}{\partial \theta^2}\right) = 0 \tag{5.76}$$

The displacement function is of the same form as for axial compression. Introducing Equation (5.55) there comes out,

$$\sigma_{\theta E} = -\frac{pr}{t} = \frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{l}\right)^2\left[\frac{\left(1+\bar{n}^2\right)^2}{\bar{n}^2} + \frac{12Z^2}{\pi^4\bar{n}^2\left(1+\bar{n}^2\right)^2}\right] \tag{5.77}$$

where one axial wave (m=1) always gives the lowest buckling load. The last term is interpreted as the buckling coefficient, C_θ. The smallest value of C_θ may be determined by trial. If $\bar{n}$ is assumed large (>>1), analytical minimization of Equation (5.77) gives

$$C_\theta^{(2)} = \frac{4\sqrt{6}}{3\pi}\cdot\sqrt{z} \tag{5.78}$$

The buckling stress becomes

$$\sigma_{\theta E} = -\frac{pr}{t} = \frac{\pi\sqrt{6}}{3\left(1-\nu^2\right)^{3/4}}\frac{Et^{3/2}}{\sqrt{r}}\frac{1}{l} \tag{5.79}$$

The expression shows that the buckling stress is inversely proportional with the ring spacing l, so it always pays off to reduce ring spacing as concerns the resistance against local hoop buckling.

For small ring spacing the elastic buckling stress is inversely proportional with the ring spacing squared, i.e. when "the plate" contribution to the buckling coefficient starts to become significant.. For perfect shells this effect starts to take place for Z < 60. For a cylinder with radius 5 m and thickness 20 mm, this implies ring spacing less than 2.5 m.
When imperfections are taken into account the effect of stiffening moves to higher Z values, roughly speaking Z = 240, implying a ring spacing in the range of the cylinder radius, i.e. 5 m.

For short cylinders z → 0 and the number of half wave n = πr/ℓ yielding
$\bar{n} = 1$ and hence $C_\theta^{(z=0)} = 4$. This is identical to the buckling coefficient for a long plate.

The approximate buckling coefficient valid for small and medium values of Z iis now derived from

$$\left(\frac{4}{C_\theta}\right)^2+\left(\frac{\left(4\sqrt{6Z}\right)/\left(3\pi\right)}{C_\theta}\right)^2=1 \tag{5.80}$$

and

$$C_\theta=4\sqrt{1+\frac{2Z}{3\pi^2}} \tag{5.81}$$

The first term is identical to the buckling coefficient of a plane, long plate.

When *l/r* approaches infinity, the last term Equation (5.77) reduces to

$$\sigma_{\theta E}=\frac{n^2E}{12\left(1-\nu^2\right)}\left(\frac{t}{r}\right)^2=0.275E\left(\frac{t}{r}\right)^2 \tag{5.82}$$

Long cylinders fail by ovalization for which n = 2. The value predicted with n =2 is, however, somewhat too high. A better approximation is obtained by substituting

$$\frac{12}{n^2}=4 \qquad \Rightarrow n=\sqrt{3} \tag{5.83}$$

into Equation 5.82.

For a closed cylinder, the end pressure must be included, where

$$N_x=\frac{1}{2}pr=\frac{1}{2}N_\theta \tag{5.84}$$

By retaining the term with N_x in Equation (5.8), it can be shown that Equation (5.77) still applies with

$$C'_{\theta E}=\frac{\left(1+\overline{n}^2\right)^2}{\left(\overline{n}^2+\frac{1}{2}\right)}+\frac{12Z^2}{\pi^4\left(1+\overline{n}^2\right)^2\left(\overline{n}^2+\frac{1}{2}\right)} \tag{5.85}$$

For <u>plane plates</u> (r→∞), Z = 0 and $\overline{n}$ = 0, therefore

$$C_{\theta E}^{(z=0)}=2 \tag{5.86}$$

The coefficient is equal to 2 and not 1 because the buckling resistance is referring to the circumferential stress and not the axial stress.

For <u>very curved panels</u> (Z >>1), $\overline{n}^2$ >>1 and Equation (5.82) can be written as,

$$C'_{\theta E}\approx\left[\overline{n}^2+\frac{12Z^2}{\pi^4}\frac{1}{\overline{n}^6}\right] \tag{5.87}$$

Minimizing $C'_{\theta E}$ 5 with respect to $\bar{n}$, we obtain

$$\bar{n}^2 = \frac{\sqrt{6Z}}{\pi} \qquad \Rightarrow C'_{\theta E} = \frac{4\sqrt{6Z}}{3\pi} \tag{5.88}$$

which is identical to value obtained for an open cylinder; the effect of the transverse (axial) stress disappears for very curved panels.

The approximate buckling coefficient valid over the entire Z range reads,

$$C'_{\theta E} = 2\sqrt{1 + 8\frac{Z}{3\pi^2}} \tag{5.89}$$

The various buckling coefficients are shown in Figure 5.12.

The equivalent stress intensity is often used as a parameter in the plasticity correction. When the end pressure is included this becomes,

$$\sigma_{eq} = \sqrt{\sigma_x^2 + \sigma_\theta^2 - \sigma_x\sigma_\theta} = \frac{1}{2}\sqrt{3}\sigma_\theta \tag{5.90}$$

Curved panels may be analyzed in the same manner as shown in the previous section, (see Equation (5.73)).

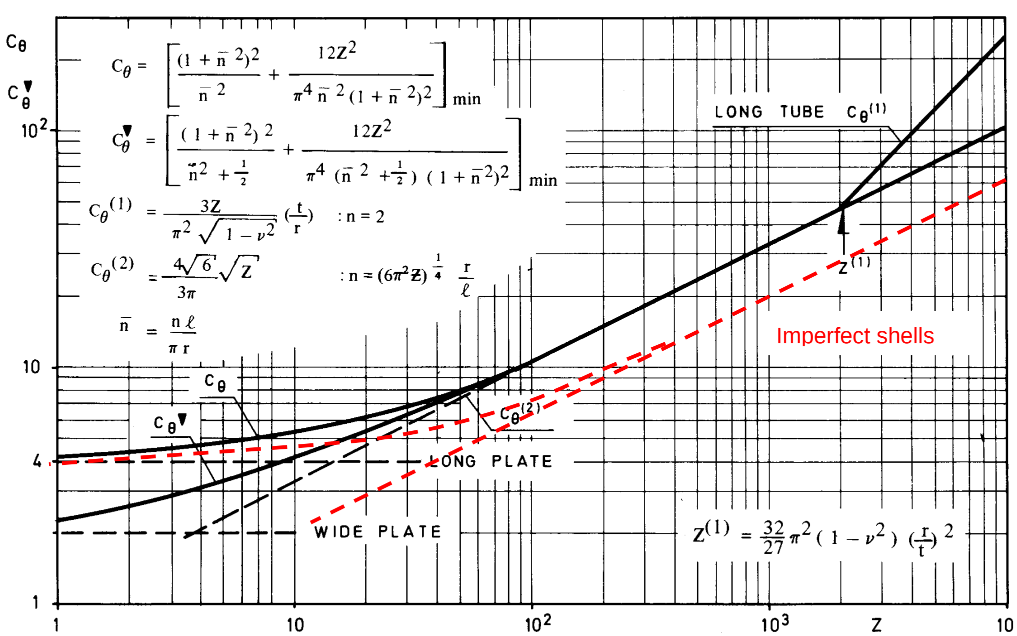


Figure 5.12 The Buckling Coefficient for External Pressure, /5.2/.

5.4.6 Torsion

For a cylindrical shell subjected to a twisting moment about its longitudinal axis, Equation (5.8) simplifies to

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} - \frac{2}{r}N_{x\theta}\nabla^4\frac{\partial^2 w}{\partial x\partial\theta} = 0 \tag{5.91}$$

Under torsional loading, the deflection pattern consists of a number of waves that spiral around the cylinder from one end to the other. For a long cylinder this may be represented by a function of the form

$$w = \delta\sin\left(\frac{m\pi r}{\ell}x - n\theta\right) \tag{5.92}$$

Introduction of Equation (5.90) into Equation (5.89) gives

$$\sigma_{x\theta,E} = \frac{E}{12\left(1-\nu^2\right)}\left(\frac{t}{r}\right)^2\left[\frac{\left(\overline{m}^2+n^2\right)^2}{2\overline{m}n} + \frac{6\overline{m}^3}{\left(\overline{m}^2+n^2\right)^2 n}\left(\frac{r}{t}\right)^2\left(1-\nu^2\right)\right] \tag{5.93}$$

where $\overline{m} = m\pi r/l$.

For long cylinders, the shell buckles in two circumferential waves, i.e. n=2. It may also be assumed that $\overline{m} << 4$. Introducing this approximation and minimizing Equation (5.91), there comes out

$$\sigma_{x\theta,E} = \frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{l}\right)^2\left[\frac{\sqrt{6}+\sqrt{2/3}}{\pi^2}\frac{Z}{\left(1-\nu\right)^{3/4}}\sqrt{\frac{t}{r}}\right] = \frac{0.272}{\left(1-\nu\right)^{3/4}}\left(\frac{t}{r}\right)^{3/2} \tag{5.94}$$

Donnel's theory is inaccurate for small number of circumferential waves (n=2). Therefore, the factor 0.272 in Equation (5.92) should be replaced by 0.236. For shorter cylinders the boundary conditions can no longer be disregarded. A solution can be obtained by use of a deflection function composed of a finite sum of terms. According to Timoshenko and Gere, /5.6/, the following buckling coefficient applies

$$C_\theta = 5.34\sqrt{1+0.02572Z^{3/2}} \tag{5.95}$$

For large Z, Equation (5.94) can be simplified to,

$$\sigma_{x\theta,E} = \frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{l}\right)^2\left[0.856Z^{3/4}\right] = 0.735\frac{E}{Z^{1/4}}\frac{t}{r} \tag{5.96}$$

The buckling coefficient for curved panels subjected to shear may be obtained from Figure 5.13.

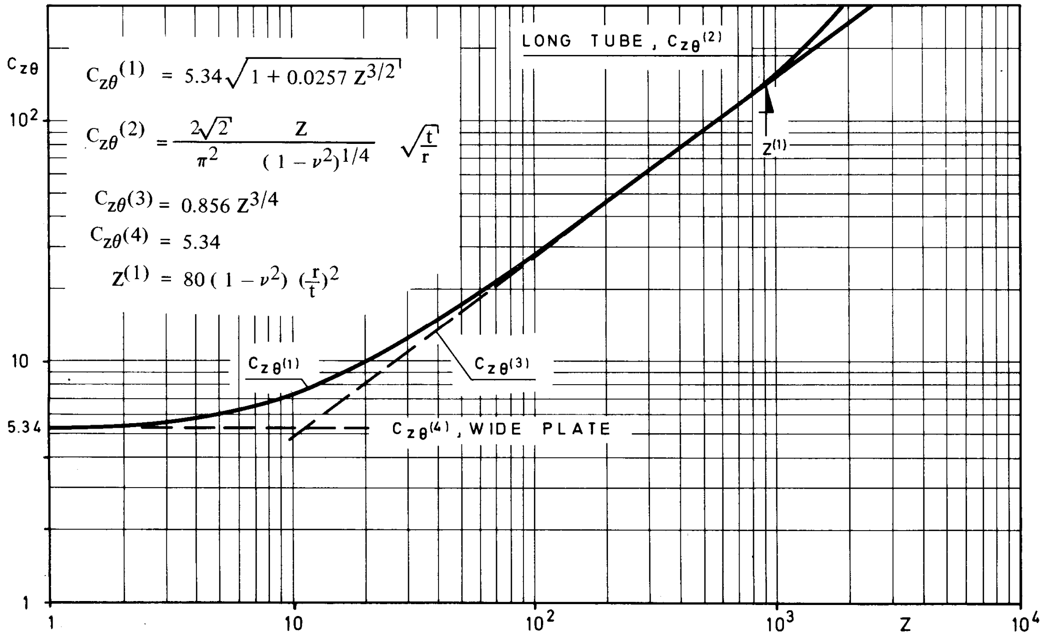


Figure 5.13 The Buckling Coefficient for Torsion, /5.2/.

5.5 Buckling of Imperfect Cylindrical Shells

5.5.1 General

The classical theory for buckling of cylindrical shells is valid only for idealized structures. Particularly, two effects have a detrimental effect on the real buckling load of welded cylindrical shells:-

- material imperfections,
- shape imperfections.

Material imperfections, like residual stresses and heterogeneities, mainly affect the buckling load in the elasto-plastic range, and therefore their effect is taken into account in the plasticity reduction factor, $\varphi = \varphi(\bar{\lambda})$. *Shape imperfections* are important both in the elastic and elastic-plastic range. Hence, their effect is accounted for by reducing the classical buckling strength.

Figure 5.12 shows, in principle, a sketch of experimental data plotted versus the reduced slenderness ratio

$$\bar{\lambda} = \sqrt{\frac{\sigma_Y}{\sigma_E}} \qquad (5.97)$$

As commented upon, in Section 5.1, the test results show a significant scatter, which could be represented by a distribution function as indicated in Figure 5.14. Ideally, a design curve should be determined so as to represent some lower bound to the test results, say the 5% percentile in the distribution function. Of course, it is not possible to carry out such an analysis accurately, but this is the ultimate goal of the modification of the classical buckling loads due to shape and material imperfections.

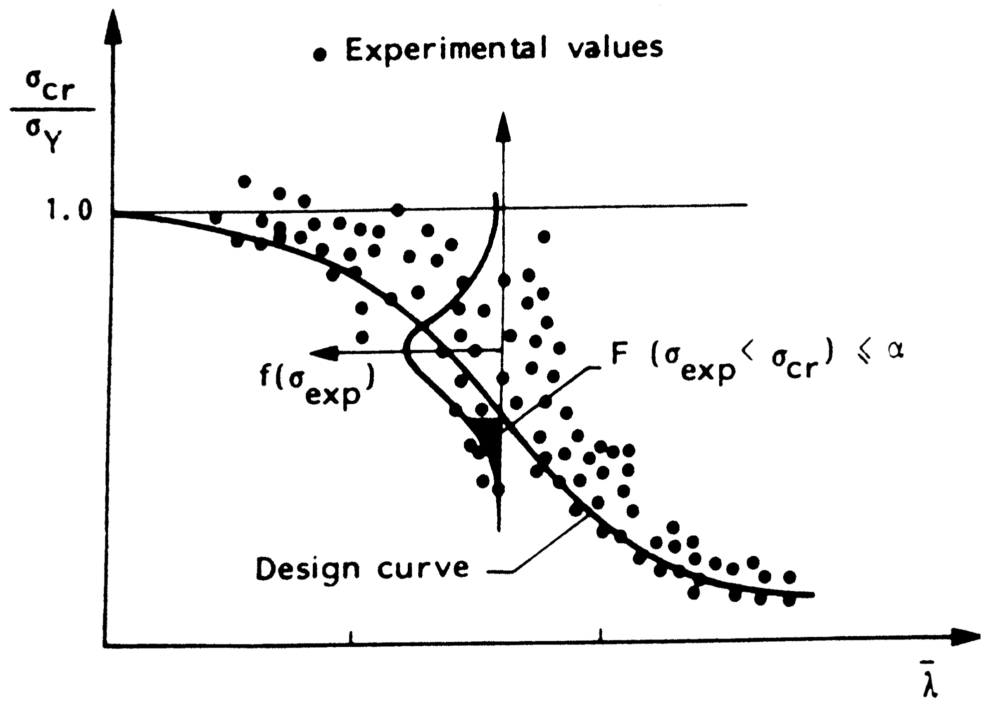


Figure 5.14 Principle Illustration of Experimental Data For Shell Buckling.

5.5.2 Shape Imperfections

The effect of shape imperfections is conveniently accounted for by modifying the classical elastic buckling resistance, σ_{cl}, as derived in Section 5.2, by an empirical reduction, or knock-down factor ρ.

$$\sigma_E = \rho \sigma_{cl} \tag{5.98}$$

Considering the *combined* buckling coefficients developed in the preceding sections, e.g. Equations (5.66, 5.73, and 5.78), it is observed that they can all be represented by the general expression,

$$C = \psi \sqrt{1 + \left(\frac{\xi}{\psi}\right)^2} \tag{5.99}$$

where ψ is the plate buckling coefficient for a plane plate, and ξ is the contribution from the curved shell (asymptote for large Batdorf parameter (Z) values).

In Chapter 7, it was shown that plates are little sensitive to shape imperfections. Hence, the knock-down factor should be applied to the curved shell contribution. The buckling coefficient is therefore modified as follows,

$$C = \psi \sqrt{1 + \left(\frac{\rho \xi}{\psi} \right)^2} \tag{5.100}$$

The effect of the imperfection factor is illustrated in
Figure **5.15**. Various formulas exist for the knock-down factors. For axial compression and bending, DNV /5.1/ uses

$$\rho = \begin{cases} \dfrac{0.5}{\sqrt{1 + \dfrac{r/t}{150}}} & \text{– axial compression} \\ \dfrac{0.5}{\sqrt{1 + \dfrac{r/t}{300}}} & \text{-bending} \end{cases} \tag{5.101}$$

These imperfection factors are plotted in
Figure **5.16** along with ECCS recommendations. For external pressure, torsion, and shear, a factor $\rho = 0.6$ is simply used, /5.1/. For other types of loading the buckling load is much less influenced by imperfections.

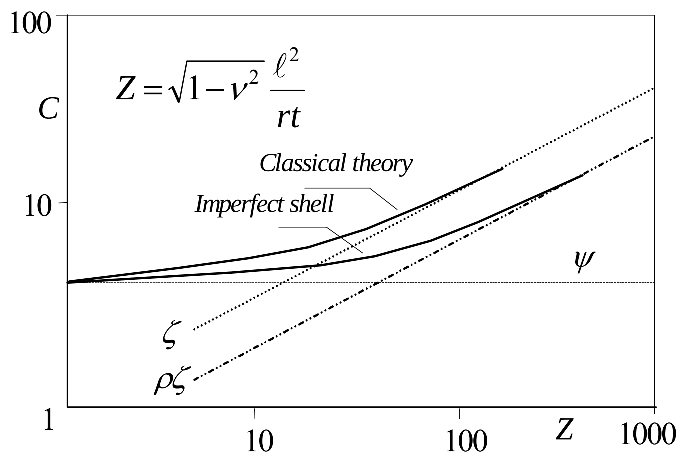


Figure 5.15 The Influence of Shape Imperfection on Buckling Coefficient.

Table 5-2 Buckling Coefficients For Unstiffened Curved Panels, Mode a) Shell Buckling

	ψ	ξ	ρ
Axial Stress	4	$0.702 Z_S$	$0.5\left(1+\dfrac{r}{150t}\right)^{-0.5}$
Shear Stress	$5.34 + 4\left(\dfrac{s}{l}\right)^2$	$0.856\sqrt{\dfrac{s}{l}}Z_S^{3/4}$	0.6

Circumfrencial Compression	$\left[1+\left(\frac{s}{l}\right)^2\right]^2$	$1.04\frac{s}{l}\sqrt{Z_S}$	0.6

Table 5-3 Buckling Coefficients For Unstiffened Cylindrical Shells, Mode a) Shell Buckling

	ψ	ξ	ρ
Axial Stress	1	$0.702\,Z$	$0.5\left(1+\frac{r}{150t}\right)^{-0.5}$
Bending	1	$0.702\,Z$	$0.5\left(1+\frac{r}{300t}\right)^{-0.5}$
Torsion and Shear force	5.34	$0.856Z^{3/4}$	0.6
Lateral Pressure	4	$1.04\sqrt{Z}$	0.6
Hydrostatic Pressure	2	$1.04\sqrt{Z}$	0.6

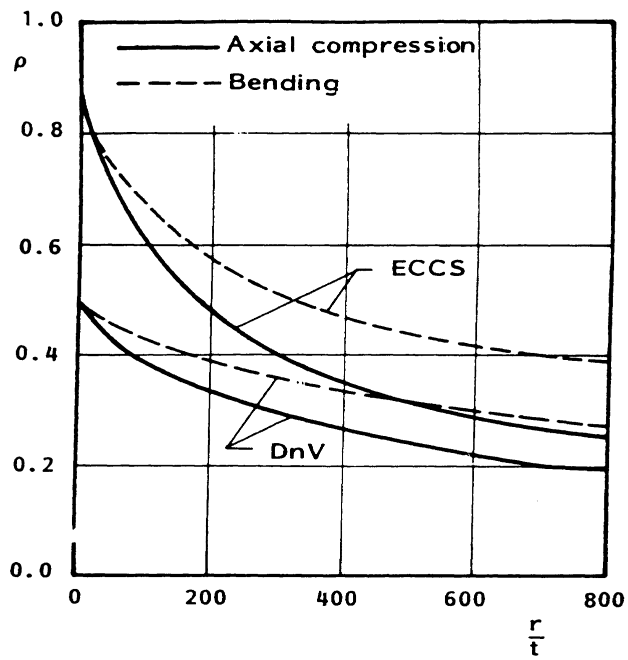


Figure 5.16 Imperfection Factors for Cylinders Subjected to Axial Compression and Bending.

5.6 Buckling Coefficients

5.6.1 Elasto-Plastic Buckling

Several methods are available for modifying the elastic critical stress due to plasticity. As is the case for plated structures, the ϕ-method is often used in connection with offshore shell structures. The critical load is defined by

$$\sigma_{cr} = \varphi \sigma_Y \tag{5.102}$$

where ϕ is a function of the reduced slenderness ratio, $\bar{\lambda}$. A widely used expression is the Merchant- Rankine formula. This is based on the following interaction function,

$$\left(\frac{\sigma_{cr}}{\sigma_E}\right)^2 + \left(\frac{\sigma_{cr}}{\sigma_Y}\right)^2 = 1 \qquad (5.103)$$

It is seen that the critical stress approaches asymptotically the Euler buckling stress for slender structures and the yield stress for stocky members. Solving for the critical stress we get,

$$\sigma_{cr} = \frac{1}{\sqrt{1+\bar{\lambda}^4}}\sigma_Y \qquad (5.104)$$

In Figure 5.17, Equation (5.102) is compared with various other ϕ-relations.

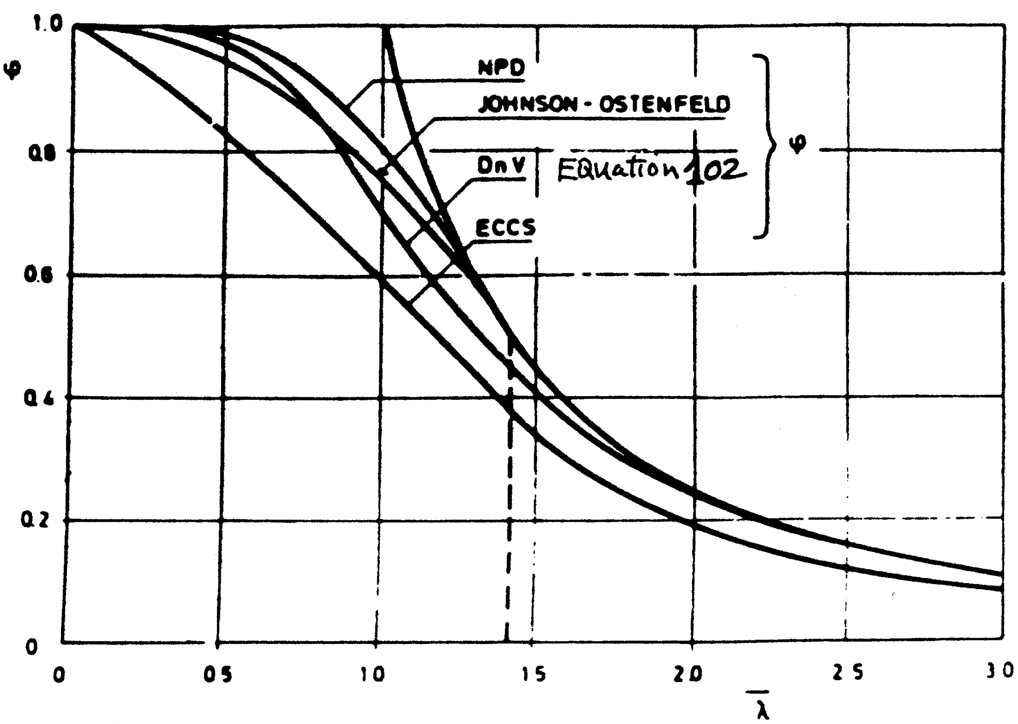


Figure 5.17 Various Design Curves For Shell Buckling.

5.6.2 Combined Loading

Analytical treatment of buckling under combined loading is generally complicated. A convenient technique is to use formulas of the interaction type. Formally, for N load components this may be written,

$$\sum_{i=1}^{N}\left(\frac{\sigma_i}{\sigma_{cr}^i}\right)^{\gamma_i}=1 \tag{5.105}$$

where σ_{cr}^i is the critical load (accounting for possible plasticity effects) when σ_i acts alone. The exponents, γ_i, are partly supported by theoretical considerations and partly verified by experiments. For a structure subjected to axial compression, bending, external pressure, torsion, and shear, the following formula has been suggested /5.2/

$$\left(\frac{\sigma_x}{\sigma_{xcr}}+\frac{\sigma_b}{\sigma_{bcr}}\right)^2+\left(\frac{\sigma_\theta}{\sigma_{\theta cr}}\right)^2+\left(\frac{\sigma_{x\theta}}{\sigma_{x\theta cr}}\right)^2=1 \tag{5.106}$$

An alternative approach is to start with an interaction formula for elastic buckling (substitute σ_{cr}^i by σ_E^i in Equation (5.104)) assume proportional loading, calculate the equivalent stress and modify for plasticity. Assuming a linear interaction relation (γ_i =1), the following expression emerges for the load case considered above

$$\frac{\sigma_x}{\sigma_{xE}}+\frac{\sigma_b}{\sigma_{bE}}+\frac{\sigma_\theta}{\sigma_{\theta E}}+\frac{\sigma_{x\theta}}{\sigma_{x\theta E}}=\frac{\sigma_{eq}}{\sigma_{eqE}} \tag{5.107}$$

where the equivalent stress according to von-Mises is

$$\sigma_{eq}=\sqrt{(\sigma_x+\sigma_b)^2-(\sigma_x+\sigma_b)\sigma_\theta+\sigma_\theta^2+3\sigma_{x\theta}^2} \tag{5.108}$$

The equivalent reduced slenderness ratio, $\bar{\lambda}_{eq}$, is obtained from

$$\bar{\lambda}_{eq}^2=\frac{\sigma_Y}{\sigma_{eqE}}=\frac{\sigma_Y}{\sigma_{eq}}\left(\frac{\sigma_x}{\sigma_{xE}}+\frac{\sigma_b}{\sigma_{bE}}+\frac{\sigma_\theta}{\sigma_{\theta E}}+\frac{\sigma_{x\theta}}{\sigma_{x\theta E}}\right) \tag{5.109}$$

The equivalent buckling resistance can then be determined by Equation (5.102).

5.7 Buckling of Longitudinally Stiffened Shells

5.7.1 General

Longitudinally, stiffened shells may be divided into three categories /5.80/, see Figure 5.18.

- Category A includes cylinders with few stiffeners. The shell behaves basically like an unstiffened shell. The only effect of the stiffeners is to increase the total cross-sectional area and the moment of inertia of the cylinders.
- Category B includes cylinders with closely spaced heavy stiffeners. In this case the effect of curvature is often neglected and the shell is modelled as an equivalent stiffened plane panel.

- <u>Category C</u> includes cylinders with closely spaced light stiffeners. By using the smeared stiffener technique, the shell may be assumed to act like an orthotropic shell.

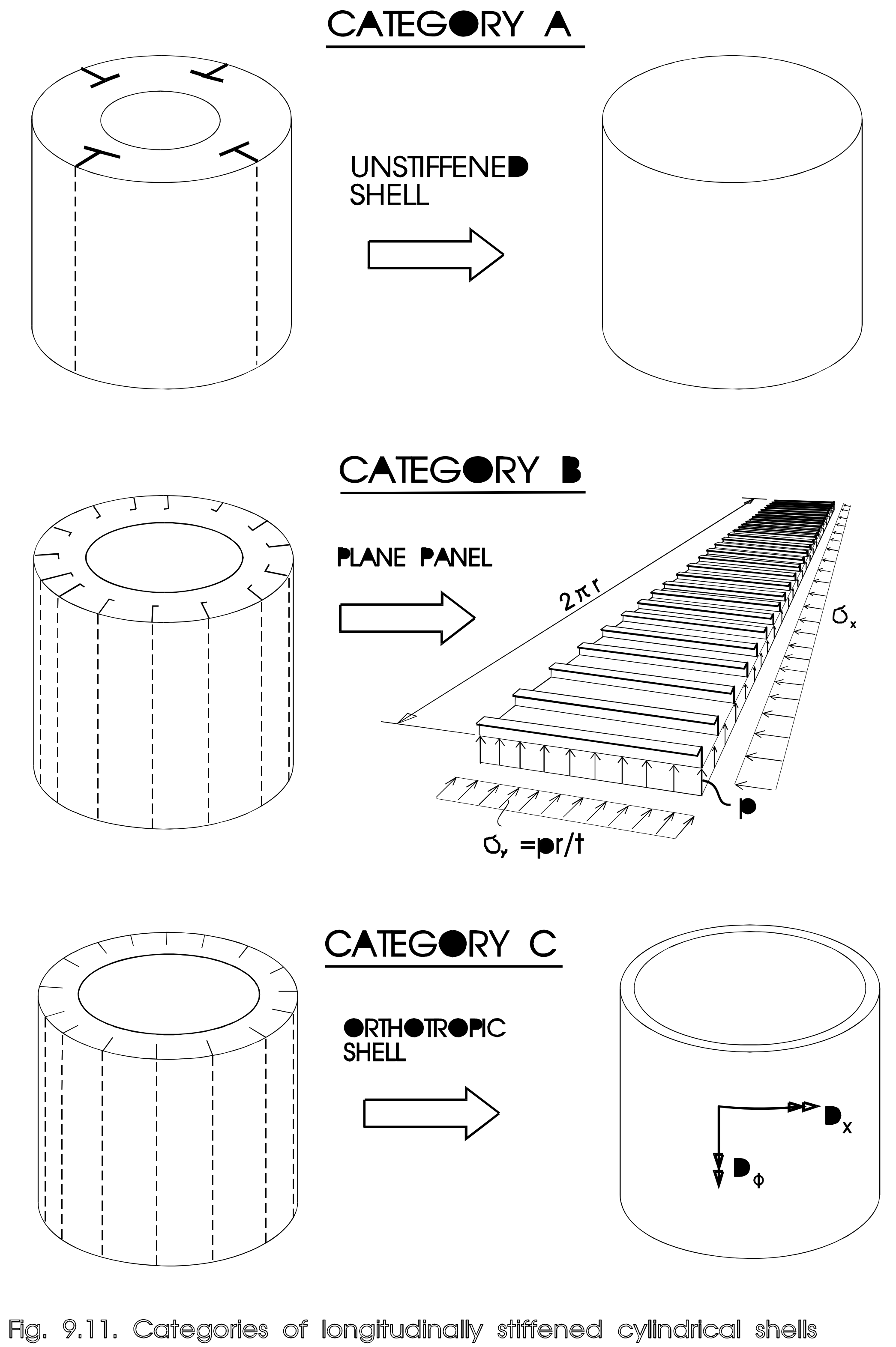


Figure 5.18 Categories of Longitudinally Stiffened Cylindrical Shells.

The criterion for subdivision into categories is set somewhat arbitrary /5.14/, (cfr. Figure 5.11). For

$$Z_S \geq 9 \qquad \text{i.e.} \frac{s}{t} > 3\sqrt{\frac{r}{t}} \qquad \text{apply category A}$$

and,

$$Z_S \leq 9 \qquad \text{i.e.} \frac{s}{t} < 3\sqrt{\frac{r}{t}} \qquad \text{apply category B or C}$$

5.7.2 Orthotropic Shell Theory

This approach relates to shells of category C. The effect of longitudinal stiffeners can be included by adding the term

$$\frac{EI_{lef}}{s}\frac{\partial^4 w}{\partial x^4} \tag{5.110}$$

in Equation (5.3) where the term EI_{lef}/s represents the bending stiffness of a stiffener, including effective flange, equally distributed over the stiffener width. In Equation (5.8) this term becomes,

$$\frac{EI_{lef}}{s}\cdot\nabla^4\frac{\partial^4 w}{\partial x^4} \tag{5.111}$$

5.7.2.1 Axial Compression

The solution to the differential equation for axial compression becomes, (cfr. Equation (5.56))

$$\sigma_{xE} = \frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{l}\right)^2\left[\frac{1}{1+\left(A/s_e t\right)}\left(m^2\gamma + \frac{\left(m^2+\overline{n}^2\right)^2}{m^2} + \frac{12Z^2}{\pi^4}\frac{m^2}{\left(m^2+\overline{n}^2\right)^2}\right)\right] \tag{5.112}$$

where A is the ratio of the stiffener exclusive of the effective plate flange, and

$$\gamma = \frac{12\left(1-\nu^2\right)I_{lef}}{st^3} \tag{5.113}$$

defines the ratio between the stiffness of stiffeners and the shell plating. Most shells will buckle in one wave between the ends. Hence, minimizing Equation (5.110) subject to the constraint m=1, there comes out

$$C_x = \frac{1}{1+\frac{A}{s_e t}}\left(\gamma + \frac{4\sqrt{3}}{\pi^2} Z\right) \tag{5.114}$$

The last term represents the effect of shell plating alone, while the first term is the contribution from the stiffeners. For very short cylinders, the last term in Equation (5.110) becomes negligible and the smallest buckling load is obtained with $n = 0$,

$$C_{x'} = \frac{1+\gamma}{1+\frac{A}{st}} \tag{5.115}$$

This is the buckling coefficient for an orthotropic plate; refer *Grillage Buckling* in Chapter 3. Buckling of Stiffened Plates. For very large Z, the effect of stiffeners becomes small. A convenient representation of the buckling coefficient is found by applying the asymptotic approximation,

$$\left(\frac{C_{x'}}{C}\right)^2 + \left(\frac{\rho\left(4\sqrt{3}/\pi^2\right)}{C} Z\right)^2 = 1 \tag{5.116}$$

or,

$$\left(\frac{C_{x'}}{C}\right)^2 + \left(\frac{\rho\left(4\sqrt{3}/\pi^2\right)}{C} Z\right)^2 = 1 \tag{5.117}$$

The knock-down factor assumed by DNV /5.1/ is $\rho = 0.5$.

5.7.2.2 *External Lateral Pressure*

The elastic buckling load for external lateral pressure is expressed by ($m = 1$), Ref. Eq. 5.76.

$$\sigma_{\theta,E} = -\frac{pr}{t} = \frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{l}\right)^2 \left[\frac{\gamma}{\overline{n}^2} + \frac{\left(1+\overline{n}^2\right)^2}{\overline{n}^2} + \frac{12Z^2}{\pi^4 \overline{n}^2 \left(1+\overline{n}^2\right)^2}\right] \tag{5.118}$$

The minimum load is found by minimization with respect to $\overline{n}$ 6. Following the approach outlined in Section 5.4.4, the buckling coefficient for a very short cylinder ($Z = 0$) is,

$$C_{\theta'} = 2\left(1+\sqrt{1+\gamma}\right) \tag{5.119}$$

while for very long cylinders

$$C_{\theta''} = \frac{4\sqrt{6}}{3\pi}\sqrt{Z} \tag{5.120}$$

Thus, the asymptotic expression takes the form,

$$C_\theta = C_{\theta'}\sqrt{1+\left(\rho\frac{1.04\sqrt{Z}}{C_{\theta'}}\right)^2} \tag{5.121}$$

A knock-down factor, ρ = 0.6, is typically used. Buckling coefficients for panel stiffener buckling are listed in Table 5-4.

Table 5-4 Buckling Coefficients For Longitudinally Stiffened Panels

	ψ	ξ	ρ
Axial Stress	$\frac{1+\gamma_S}{1+\left(A/s_e t\right)}$	$0.702Z$	0.5
Torsion and Shear Force	$5.34+1.82\left(\frac{\ell}{s}\right)^{4/3}\gamma_S$	$0.856Z^{3/4}$	0.6
Lateral Pressure	$2\left(1+\sqrt{1+\gamma_S}\right)$	$1.04\sqrt{Z}$	0.6

5.7.2.3 Beam on Elastic Foundation

The shell buckling problem bears considerable similarity with buckling of a beam on an elastic foundation, (see Section 5.3.2.4 in T.H. Søreide, *Ultimate load analysis of marine structures*). Depending on the stiffness of the foundation, the beam may buckle in several waves. Here, a buckling mode with one wave (m = 1) is considered.

The differential equation for such a beam with a sinusoidal initial imperfection,

$$w_o = \delta_o \sin\frac{\pi x}{\ell} \tag{5.122}$$

is given by,

$$EI\frac{\partial^4 w}{\partial x^4} + N\frac{\partial^2 w}{\partial x^2} + \alpha w = N\frac{\partial^2 w_0}{\partial x^2} \tag{5.123}$$

where the axial load, N, is defined positive in compression. By applying the operator, $\partial^4/\partial x^4$, the similarity with Equation (5.8) is apparent. The solution is given by

$$N_E^* = \frac{\pi^2 EI_l}{l^2}\left(1 + \frac{\alpha l^4}{\pi^4 EI_l}\right) \tag{5.124}$$

and the critical stress is,

$$\sigma_E = \frac{N_E}{2\pi rt\left(1+\frac{A}{st}\right)} = \frac{\pi^2 EI_l}{l^2\left(st+A\right)}\left[1+\frac{\alpha l^4}{\pi^4 EI_l}\right] \tag{5.125}$$

Equation (5.123) may be rearranged so that

$$\sigma_E = \frac{\pi^2 EI_\ell}{l^2(st+A)}\left[1+\frac{4\sqrt{3}Z\,st^3}{\pi^2 12\left(1-\nu^2\right)I_l}\right] \tag{5.126}$$

Equations (5.123-4) are identical if,

$$\alpha = \frac{\pi^2 E}{\sqrt{3}\left(1-\nu^2\right)}\left(\frac{t}{l}\right)^2\frac{1}{rs} \tag{5.127}$$

and Equation (5.124) can be used to represent the behaviour of a stringer stiffened shell. The total deflected shape of the beam is given by

$$w_{tot} = w + w_o \tag{5.128}$$

The <u>effective</u> deflection is,

$$w = \frac{\frac{N}{N_E^*}}{1-\frac{N}{N_E^*}}\delta_o \sin\frac{\pi}{l}x \tag{5.129}$$

where N_E^* is given by Equation (5.122). The maximum moment becomes,

$$M_{\max} = -EI\left.\frac{\partial^2 w}{\partial x^2}\right|_{l/2} = \frac{N_E\frac{N}{N_E^*}}{1-\frac{N}{N_E^*}}\delta_o \tag{5.130}$$

where N_E is the Euler buckling load for zero foundation stiffness. The first yield occurs when

$$\sigma_Y = \frac{N}{st+A} + \frac{\frac{N}{N_E^*}}{1-\frac{N}{N_E^*}}\frac{Nw_o}{W}$$
$$= \sigma_x + \frac{\frac{\sigma_E}{\sigma_E^*}}{1-\frac{\sigma_x}{\sigma_E^*}}\frac{\sigma_x(st+A)w_o}{W} \tag{5.131}$$

The solution for the compressive stress is given by the Perry-Robertsen formula,

$$\frac{\sigma_x}{\sigma_Y} = \frac{(1+\gamma+\xi) - \sqrt{(1+\gamma+\xi)^2 - 4\gamma}}{2\gamma} \tag{5.132}$$

where,

$$\gamma = \frac{\sigma_Y}{\sigma_E^*} \quad , \quad \xi = \frac{\sigma_E}{\sigma_E^*}\frac{(st+A)w_o}{W} \tag{5.133}$$

in which W is the elastic section modulus of the stringer-shell combination.

If a reduced effective width, s_e, of the shell is assumed, the ultimate load expression reads

$$\sigma_u = \sigma_x \frac{s_e t + A}{st+A} \tag{5.134}$$

The magnitude of imperfection may be predicted by formulas similar to those specified for stiffened plates.

It should be noted that stringer stiffened cylinders are very prone to inward buckling, because otherwise circumferential stretching would occur. This means that the failure is most likely to be plate-induced for internal stiffening. As several spans may take the same deflection pattern, the stringers may experience considerable rotational restraint at the ring frames.

5.8 Buckling of Ring Stiffened Shells

5.8.1 Lateral Pressure

The buckling pressure of a ringstiffened cylinder is conveniently formulated as the sum of a shell contribution, p_c, and a ring-frame term, p_r

$$p_{cr} = p_c + p_r \tag{5.135}$$

A widely used expression is the Bryant formula /5.7/ which is based on the elastic potential energy

$$p_{cr} = E\frac{t}{r}\frac{k^4}{\left(n^2-1+\frac{1}{2}k^2\right)\left(n^2+k^2\right)^2} + \frac{\left(n^2-1\right)EI_r}{r^3 l} \tag{5.136}$$

where $k = \pi r/L$. The critical number of waves is usually in the range of 2 to 6. Sometimes the shell contribution is neglected and n is selected equal to 2 so that,

$$p_{cr} = \frac{3EI_r}{r^3 l} \tag{5.137}$$

This represents ovalization of the ring.

The magnitude of the stress predicted by Equation (5.134) is usually far above the elastic limit. Various methods exist for taking elasto-plastic effects into account. One method is parallel to the Perry-Robertson method for column buckling (see reference /5.21/). The other method represents a generalization of Equation (5.135)

$$p_{cr} = \eta_c p_c + \eta_r p_r \tag{5.138}$$

where h_c and h_r are the plasticity correction factors. Various formulas have been suggested. A simple definition is

$$\eta_c = \frac{\sqrt{E_s E_t}}{E} \quad , \quad \eta_r = \frac{E_t}{E} \tag{5.139}$$

where E_s and E_t are the secant and tangent modulus respectively.

From Equation (5.135) the requirement to ring moments of inertia can be derived. The total circumferential force between rings is given by $N_\theta = prl$.

Assuming a sinusoidal imperfection with amplitude w_o, the bending stress at the top of the free flange is

$$\sigma_\theta^b = \frac{pr\ell \cdot w_o \cdot z_t}{I_r}\frac{1}{1-\frac{1.5p}{p_{cr}}} \tag{5.140}$$

where z_t is the distance from the centroid of the top flange and p_{cr} is given by Equation (5.135). A safety factor of 1.5 is applied in the denominator of the magnification term. A conservative design criterion is to require that the total stress at the top flange be less than half the critical stress for torsional buckling of the ring, σ_T (safety factor of 2). Hence,

$$\sigma_\theta^r + \sigma_\theta^b \le \frac{1}{2}\sigma_T \tag{5.141}$$

This yields the following requirement to the moment of inertia of the ring stiffener,

$$I_r \leq \frac{pr^3 l}{3E}\left(1.5 + \frac{3Ez_t w_o}{r^2(0.5\sigma_T - \sigma_\theta^r)}\right) \tag{5.142}$$

If this requirement is not met, a more accurate calculation of the critical pressure (sketched for ring buckling below), must be performed as shown below.

This solution was developed by Bodner in 1957. It can be expressed as,

$$\sigma_E = \frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{l}\right)^2 C \tag{5.143}$$

where the buckling coefficient is

$$C = \frac{1}{(1+\alpha_r)(0.5+\bar{n}^2)}\left[\left(1+\bar{n}^2\right)^2 + \gamma_r \bar{n}^4 + \frac{12Z^2}{\pi^4}\frac{1}{\left(1+\bar{n}^2\right)^2}\right] \tag{5.144}$$

This is identical to the shell buckling coefficient in Equation (5.82), except that the smeared bending contribution from the ring-stiffeners are added

$$\gamma_r = \frac{12\left(1-\nu^2\right) I_r^e}{t^3 l} \tag{5.145}$$

where l is the ring-stiffener spacing, I_r^e is the moment of inertia of the ring-stiffener including the effective shell flange.

An approximate buckling coefficient may be obtained analogous to the approach used for unstiffened cylinder. The expression reads,

$$C = 2\frac{1+\gamma_r}{1+\alpha_r}\left[\sqrt{1+\frac{8Z}{3\pi^2\sqrt{1+\gamma_r}}} - \frac{\gamma_r}{1+\gamma_r}\right] \tag{5.146}$$

where,

$$\alpha_r = \frac{A_r}{l_{eo} t} \tag{5.147}$$

is the ratio between the ring-stiffener area and the effective shell flange area. The effective length of the cylinder is taken as the distance between the end closures, bulkheads or heavy ring frames, as indicated in Figure 5.19.

Elasto-plastic buckling is calculated by means of a first yield criterion as used for beam columns. It is assumed that the ring stiffeners have an initial out-of-roundness compatible to the buckling mode,

$$w_o(\theta) = w_o \sin n\theta \tag{5.148}$$

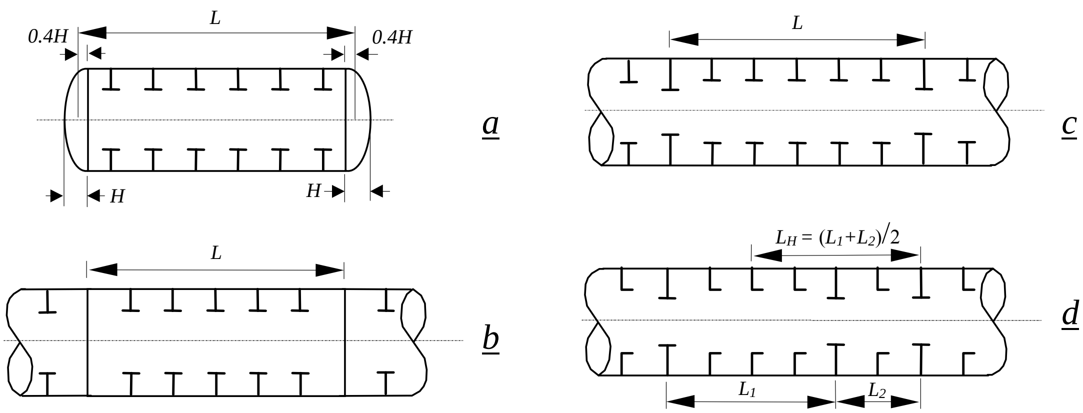


Figure 5.19 Definition of the Effective Length.

In calculating the bending contribution, it is necessary to take into account that the stress in the ring stiffener and in the shell flange is different due to restrained contraction.

The total force in between the ring frames can be expressed as

$$F_{\theta'} = pr\ell = \sigma_\theta^r tl \frac{1+\alpha}{1-0.5\nu} \tag{5.149}$$

Here, it is also taken into account that the stress at the top of the ring, σ_θ^r, is larger than the stress in the plate, so that in calculating the effective bending moment the stress is evaluated at the shell flange. This gives,

$$F_{\theta'} = \sigma_\theta^r tl \frac{r_t}{r} \frac{1+\alpha}{1-0.5\nu} \tag{5.150}$$

where r_t is the radius to the top of the ring stiffener.

In calculating the moment of inertia of the ring frame, it is assumed that the ring may have some post-buckling capacity beyond the ring failure. The apparent moment of inertia is expressed as,

$$I_r^{app} = I_r \frac{p_r + p_s}{p_r} = I_r \frac{1}{1 - \frac{p_s}{p_r + p_s}} = I_r \frac{1}{1 - \frac{C_s}{C}} \tag{5.151}$$

where p_r and p_s are assumed to be the contribution from the ring and the shell flange to the critical pressure for ring-stiffener buckling, C_s is the buckling coefficient for unstiffened shell subjected to lateral pressure, and C is given by Equation (5.144). The first yield criteria can now be established as follows,

$$\frac{\sigma_\theta^r}{\sigma_Y}+\frac{\sigma_\theta^r t l \dfrac{1+\alpha}{(1-0.5\nu)}\dfrac{r_t}{r}\left(1-\dfrac{C_s}{C}\right)z_t w_o}{\sigma_Y\left(1-\dfrac{\sigma_\theta^r}{\sigma_e}\right)I_r}=1 \tag{5.152}$$

where z_t is distance from the centroid of the stiffener (including effective shell flange) to the top of the ring flange. This equation has the solution,

$$\frac{\sigma_\theta^r}{\sigma_y}=\frac{1+\mu+\bar{\lambda}^2-\sqrt{\left(1+\mu+\bar{\lambda}^2\right)^2-4\bar{\lambda}^2}}{2\bar{\lambda}^2} \tag{5.153}$$

where the equivalent deflection parameter, μ, is defined as

$$\mu=w_o\frac{z_t}{\left(i_e^r\right)^2}\frac{r_t}{r}\frac{l}{l_{eo}}\left(1-\frac{C_s}{C}\right)\frac{1}{(1-0.5\nu)} \tag{5.154}$$

$$(i_e^r)^2=\frac{I_r}{l_{eo}t(1+\alpha)} \tag{5.155}$$

The solution is also presented in Figure 5.20 as a function of μ.

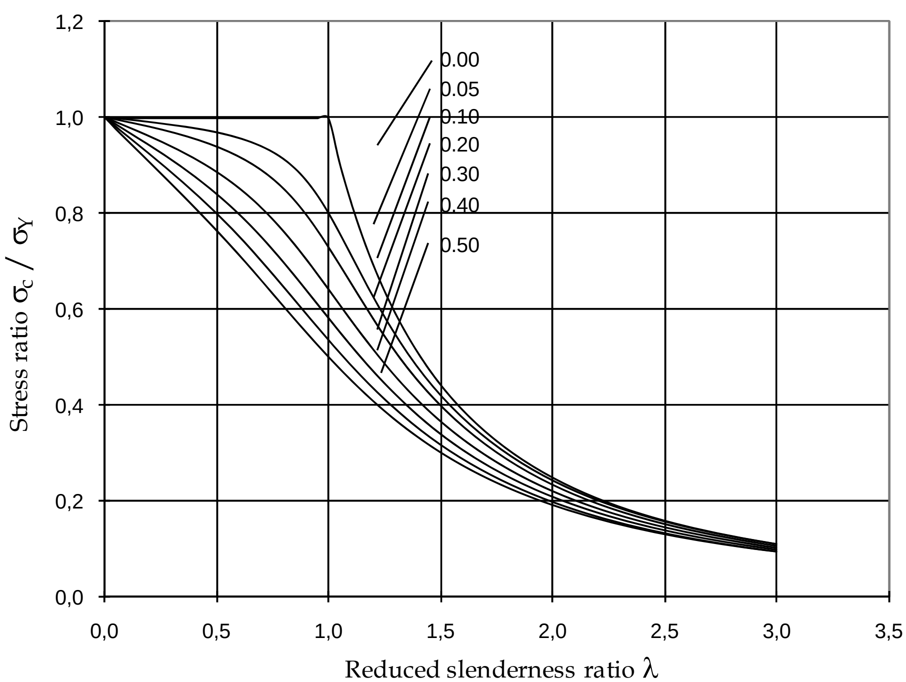


Figure 5.20 Critical Stress for Ring-Stiffener Failure.

5.8.2 Combined Loading

When the cylinder is subjected to a combination of axial compression, bending, external pressure, torsion, and shear, a linear interaction formula is often used. That is, the total ring moment of inertia is taken as the sum of the requirements for each separate load condition.

5.9 General Buckling

5.9.1 Axial Compression and Bending

As for plane panels, overall buckling is an undesired event. Hence, for offshore structures, the rings are designed so as to ensure that this failure mode is prevented. This is achieved if the general buckling resistance is higher than the buckling load of the shell and, if present, longitudinal stiffeners.

The expressions for general buckling become very complicated for analytical treatment. Therefore, considerable simplification is obtained if the plane panel analogy can be applied, neglecting the curvature effect of the shell. The following derivation is based on Section 7.5. The elastic buckling resistance for a longitudinal stiffener with associated plate flange is,

$$\sigma_{E'} = \frac{\pi^2 D}{a^2\left(t+\frac{A}{s}\right)} i_a \tag{5.156}$$

where it has been assumed that $i_a >> 1$. The resistance to general buckling is expressed by Equation (7.93),

$$\sigma_{E''} = \frac{\pi^2 D}{b^2\left(t+\frac{A}{s}\right)}\left[2\sqrt{i_a i_b}\right] \tag{5.157}$$

The requirement that $\sigma_{E''} > \sigma_{E'}$ 7 yields,

$$i_b > \left(\frac{b}{a}\right)^4 \frac{i_a}{4} \quad \xrightarrow{\text{or}} \quad I_b \ge \frac{b^4}{4\pi^2}\frac{\sigma_{E'}}{E}\frac{1}{a}\left(t+\frac{A}{s}\right) \tag{5.158}$$

If buckling is not elastic, $\sigma_{E'}$ may be substituted by the elasto-plastic buckling resistance, $\sigma_{c'r}$. This is assumed to be conservative because the transverse girders are less influenced by plasticity and shape imperfections.

While a plane panel buckles with one wave in the transverse direction several waves may be generated in the ring direction of the shell. However, Equation (5.156) may still be applied if *b* is substituted by *pr/n*, and *a* by *l*,

$$I_r \ge \frac{\pi^2}{4n^4}\left(1+\frac{A}{st}\right)\left(\frac{t}{l}\right)\frac{\sigma_{c'r}}{E} r^4 \tag{5.159}$$

Equation (5.157) may be used for both, shell with stringers and shell without stringers. A problem arises when assessing the number of circumferential waves. One possibility is to use the values predicted by elastic theory. In reference /5.1/, the term $4n^4/\pi^2$ is replaced by 500.

When calculating I_r account should be taken for reduced shell plating. For axisymmetric buckling, DNV proposes

$$l_{eff} = \min\left(1.56\sqrt{rt}\ , l\right) \tag{5.160}$$

5.9.2 Torsion and Shear

The requirement is that general buckling should be predicted by buckling in torsion and shear. The elastic general buckling resistance can be written as a generalization of the isotropic buckling expressions, (see Equation (5.93)).

$$\sigma^{G}_{x\theta,E} = \frac{\pi^2 E}{12\left(1-\nu^2\right)}\left(\frac{t}{l}\right)^2 \left(1+\gamma_r\right)k \tag{5.161}$$

where

$$k = 5.34\sqrt{1+0.0257 Z_G^{3/2}} \tag{5.162}$$

$$Z_G = \frac{L^2}{rt}\sqrt{\left(1-\nu^2\right)\frac{1+\left(A/st\right)}{1+\gamma_r}} \tag{5.163}$$

By introducing the approximations, $k \approx 0.856 Z_G^{3/4}$ and $\gamma_r >> 1$, the requirement that $\sigma^{G}_{x\theta,E} > \sigma_{x\theta,E}$, (cfr. Equation (5.94)), gives

$$I_r > \frac{0.15}{\left(1+\dfrac{A}{st}\right)^{3/5}}\left(\frac{\sigma_{\theta,cr}}{E}\right)^{8/5}\left(\frac{r}{L}\right)^{1/5} Lrtl \tag{5.164}$$

5.10 Column Buckling

Very long cylindrical shells may fail by column buckling modes which can not be described by linear Donnel's shell theory. Furthermore, very dangerous interaction between local buckling and column buckling may occur due second order effects of axial compression. Bending effects may also cause loss of stiffness due to ovalization.

For pure axial compression of a cylinder with initial out-of-straightness, the Perry-Robertson approach may be used where failure is defined when the maximum axial stress reaches the local buckling resistance. A more comprehensive description of overall failure modes can be found in reference /5.2/.

5.11 References

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