Exercise 7

Finite volume method for
2D inviscid Burgers’ equation

Due by 2014-10-10

Objective:
to get acquainted with the explicit finite volume method (FVM) for a 2D conservation law and to
train its MATLAB programming and numerical analysis.

Task:

We consider the inviscid two-dimensional Burgers’ equation:

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} + \frac{\partial u}{\partial y} = 0
\]  

(1)

for \(0 < x < 1, 0 < y < 1\) on a uniform grid of \(N \times N\) grid points, subject to the boundary conditions:

\[
u(0, y) = 1.25,\]
\[
u(1, y) = -0.75,\]
\[
u(x, 0) = 1.25 - 2x.
\]

a) Why is no boundary condition needed at the upper boundary?

b) The first order accurate upwind FVM is used for the spatial discretization and the explicit Euler
method for the time integration, where the initial solution is chosen as \(u(x, y) = 1.25 - 2x\).
Perform a stability analysis for the linearized equation in order to determine the maximum
allowable time step.

*Hint:* Determine the Fourier symbol \(\lambda = \lambda(k_1 \Delta x, k_2 \Delta y)\) of the upwind FVM applied to the
linearized equation. Derive the stability condition for \(\lambda \Delta t\) to lie in the stability domain of the
explicit Euler method by using geometrical arguments, e.g. the points \(z + re^{i\theta}, \theta \in \mathbb{R}\), lie on a
circle with radius \(r\) around point \(z \in \mathbb{C}\).

c) Write a MATLAB program which calculates the steady state solution with the method described
above. Use as a criterion to stop the time stepping process that the change in the solution from
one time level to the next is sufficiently small. The criterion should be applicable for any number
of grid points.
d) Apply the program to grids with $N = 21$, $N = 41$ and $N = 81$ grid points (i.e. $N - 1$ cells) in each spatial direction. How many time levels are needed until the stopping criterion is reached for each grid? Make a contour plot of the steady state solution in the $x$-$y$ plane for each grid. Comment your results.

e) **optional**

(i) Plot the 2-norm of the solution change $||u^n - u^{n-1}||_2$ over the time levels $n$.

(ii) Study the convergence rate of the FVM for the steady state solution.

(iii) What is the relation between the calculated steady state solution and the solution of the one-dimensional inviscid Burgers' equation?

f) **optional**

Change the first order upwind FVM into a high resolution method using the MUSCL approach and the minmod limiter. Perform grid refinement and study the convergence rate of the FVM for the steady state solution. Test both the explicit Euler method and the TVD Runge-Kutta method for time discretization. Comment your results.